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Essential Mathematics for Minimum Army Needs[1]

THAT every enlisted man in the Army has vital need for mathematics is no longer open to question. This fact was strongly indicated in the statement of an earlier committee[2] on mathematics for inductees, whose report, to be treated more fully later on, was based upon analyses of a large sample of Army and Navy training manuals. The fact of essential mathematical needs has been even more certainly established by the present committee on mathematics, which, working with the co-operation of the Civilian Pre-Induction Training Branch of the Army Service Forces and the U. S. Office of Education, makes its report in this statement.

This committee consisted of: Virgil S. Mallory, Montclair (N. J.) State Teachers College, chairman; Rolland R. Smith, Public Schools, Springfield, Mass.; C. Louis Thiele, Public Schools, Detroit, Mich.; F. Lynwood Wren, George Peabody College for Teachers; for the Civilian Pre-Induction Training Branch, A. S. F. William A. Brownell, Duke University, consultant; and for the U. S. Office of Education John Lund and Giles M. Ruch.

BASIS OF PRESENT REPORT

The procedure employed by the present committee was to confer with Army officers directly in charge of training enlisted men and to observe the basic training process itself during the first thirteen weeks of the inductee's Army life. To systematize both interviews and observations, a check list of one hundred forty-one items was employed. This check list, carefully prepared by the committee, had the benefit of criticism from more than twenty Army training officers stationed in the Washington area. It was then tried out in the original and in a revised form with a dozen training officers in nearby camps. In its final form its contents provided the basis for informal conferences with another group of ninety-six training officers and for item-by-item checking by one hundred seventy-eight officers who were serving as instructors in basic training in seventy-four different kinds of Army jobs in Replacement Training Centers and unit training centers in eight States. The centers visited are operated by branches which together train seventy-five per cent of the enlisted men of the Army. The mathematics outlined later is that which is *actually needed* by nearly all men in their

[2] This committee of the U. S. Office of Education which worked in conjunction with the Civilian Pre-Induction Training Branch, consisted of: Rolland R. Smith, chairman, Virgil S. Mallory, William D. Reeve, Giles M. Ruch, and Raleigh Schorling. The first two named, it will be noted, are members of the present committee. The report of this earlier committee will be found in: the April 1, 1943, issue of *Education for Victory*, the April, 1943, issue of the *Bulletin of Secondary-School Principals*, and in the March, 1943, issue of THE MATHEMATICS TEACHER.

basic training.[3] Even the four per cent of illiterates in the Army are now being taught many of these topics in special classes. The purpose of the committee's investigation was to determine those items in mathematics which should make up the *minimum* equipment of the inductee. Every inductee might well have *more* than this minimum; but he *cannot have less* and meet successfully the demands of basic training.

So much for the first fact: the enlisted man has certain minimum mathematical needs. The second fact, equally important and equally certain, is that the typical inductee does not have the training in mathematics which he needs. An accumulating, if distressing, body of evidence supports this second statement. When only one inductee out of four can select the correct answer from four suggested answers for, 5 is 20% of what number?; when only one in three can select the correct answer for, $7-5\frac{3}{4}$; and when only one in four can select the correct answer for, $.32 \div .64$; under these conditions it is clear that the inductee is ill prepared to cope with the quantitative situations he will encounter in his basic training in the Army.[4] It could be argued that these mathematical deficiencies are not new, that indeed many adults and high school students have for years had limited proficiency in mathematics, and that the war has served only to highlight the evils of a long-standing condition. It could be argued further that, had Army inductees been taught their mathematics in the elementary school and in the junior high school as mathematics should be taught, the present situation would not be so critical, and the task of preparing inductees for Army service would be relatively easy. All these arguments can be conceded. At the same time, these arguments do not lessen, rather they enhance the urgency of doing something positive and of doing it immediately. As will be made amply clear, this task is not a simple one.

It is in this spirit that the committee reports its findings and its recommendations. The findings appear in the list of mathematical items beginning on page 247. The recommendations appear, partly in what follows at once, and more completely in the sections which follow the list of items. To look ahead a bit, these later recommendations have to do (1) with the improvement of instruction in mathematics, particularly in connection with the topics listed, and (2) with suggestions for setting up a program in the public schools and elsewhere.

Nature of the Essential Mathematics

Before the list of mathematical topics is presented, it is well to state at once what this list *is* and what it *is not.*

(1) The list does *not* constitute a course in mathematics. Neither the Army nor this committee wishes to suggest a course of study to school officials and teachers. It is true that *some* kind of course, lasting if possible a full year or more, is needed to give young men about to enter the Army the basic mathematics which they will need. The responsibility for determining the nature and details of this course rests with local school authorities.

(2) To repeat what has already been said, the topics listed here represent a *minimum* of mathematical equipment for inductees. There is no thought that only the items listed should be taught, and that these should be taught only to the limits suggested. Nor is there any thought of asking capable students to substitute a course based on the listed items for the sequential mathematics of the senior high

[3] It should be emphasized that many enlisted men go on for specialized training which calls for more mathematics than is outlined in this report. These men can profit by as much of the sequential work in mathematics as they can master.

[4] These data were furnished by the Personnel Research Section of the Classification and Replacement Branch of The Adjutant General's Office. The poor showing cannot be dismissed on the ground that these inductees had merely "forgotten" their mathematics because they had not "used" it for some years.

school. For those not taking such courses, instruction in the essential mathematics listed should be carried as far as time permits. And there are other mathematical concepts and skills, not found to be part of the necessary minimum equipment, but almost certainly valuable in the Army, on which instruction might well be offered.

(3) This report is an extension of the report of the earlier U. S. Office of Education Mathematics Committee. The present report *supplements* the first one by amplifying the suggestions offered for the *lower* levels of mathematics as represented in the Special One-Year Course.

(4) The topics are appropriate objectives of instruction for all young men about to enter the Army, regardless of whether they are in school or have left, and, if in school, regardless of the grade in which they may be found. For this reason the list should supply guidance for officials and teachers of special classes (night classes, extension courses) as well as for officials and teachers in the schools.

(5) It is a fallacy to assume that enrollment in advanced high school courses in mathematics assures proficiency in the minimum essentials listed in this report. Ample evidence in the Army and in civilian life shows that the study of algebra and geometry does not guarantee the maintenance of lower-order mathematical abilities and knowledge. On this account, school officials and teachers should take steps to see whether students taking advanced mathematics possess the needed minimum essentials and to teach whatever may be necessary. It should be noted in this connection that the earlier report already referred to suggested changes in the sequential courses to give time for much of the material of the present report.

(6) This report is not without significance for instruction in mathematics in the elementary school. To the extent that the proposed special program is made necessary for deficiencies in earlier learning, it will be made unnecessary as soon as instruction at lower points in the grades is improved. Teachers of the elementary grades are therefore urged to study carefully the recommendations in the section beginning on page 247.

(7) This report does not sacrifice general education to the particular needs of the Army. Every mathematical item in the list given can be justified in terms of general education.

(8) The casual reader of this report can easily oversimplify the educational problem with respect to the needed program of instruction. He will certainly do so if he disregards the evidence and assumes that *his* students possess the minimum mathematical essentials recommended. He will also certainly oversimplify the problem if he thinks of the essentials listed purely in terms of mechanical, paper-and-pencil computation. Proficiency in computation is of course necessary, but it should not be mere mechanical proficiency. Much more than computational competence is called for.

On this last point the testimony of training officers is unequivocal. They say that many enlisted men, even those who seem to be able to obtain correct answers in abstract computation, are unable to *think* quantitatively. That is to say, they cannot use in practical situations even the limited skills which they possess.

The implications of this charge (and it was universal) are unmistakable, and they cannot be disregarded. What is needed is a reorientation, a change of emphasis, in instruction. Computation has too often been stressed, and accurate, skilled thinking in concrete quantitative situations has been minimized. Many students have acquired tricks with numbers which have proven valueless under conditions of use. Meanwhile two aspects of mathematical learning have suffered, namely, (a) understanding and (b) experience in application. An inductee *understands* when he has acquired the meanings which give system, order, and logic to mathematics,—when, for example, he knows the functions of the fundamental operations and knows when to use them and why they affect numbers as they do. He has had the requisite *experience in application* when his learning has included many occasions in which he has successfully put to use his mathematical training in situations which are significant to him.

It is hoped that this report may help to establish correct balance among the various aspects of instruction; and it is for this reason that half the space in this statement has been reserved for suggestions respecting instruction on the listed mathematical essentials.

(9) It follows from all that has been said that refresher courses in the high school and elsewhere are not the sole remedy for the present emergency. Such courses may only restore former skills which are inadequate for Army needs and for the demands of civilian life. Instead, young men about to enter the Army must be taught something which heretofore has not often enough been taught, namely, the ability to meet quantitative problems effectively, confidently, and sensibly. They must be able (a) to identify the quantitative aspects of the situations which confront them, (b) to deal with these situations by approximation and estimation when computation is not required, (c) to recognize and use the simpler symbolism of mathematics, (d) to tell when and how mathematical symbolism, concepts, and processes are to be employed, and (e) to compute accurately, quickly, and intelligently when computation is called for. Courses with objectives less ambitious than these are of limited value.

Findings with Respect to Mathematical Needs

Some explanation is required for the check list which was used by members of the committee in their visits to Army camps. The items dealt with mathematical skills, concepts, and relationships as well as with memorized formulas and rules; they covered the fields of arithmetic, and the simpler aspects of algebra, geometry, and trigonometry. With many of the items an example was given to set a working level of difficulty. Only the uses of mathematics in the thirteen weeks of basic training given to all inductees were to be recorded. The needs of candidates for officer training, and of specialists whose requirements in mathematics might be considerably more than those outlined in the check list, were not included in the investigation.

A total of one hundred seventy-eight instructors actually engaged in teaching inductees filled out the check list. They were asked to do four things: (1) to mark each item as of *frequent*, *occasional*, or *rare* use; (2) to classify the examples given as of *less*, the *same*, or *greater* difficulty than inductees would actually need; (3) to furnish practical examples of how the mathematical items listed are used; and (4) to report on mathematical needs not covered in the list. Each instructor was advised before marking the check list to be concerned only with the mathematical needs in his own particular field of instruction, not with his philosophy about needs in general.

The results of the marking of the check list are summarized, examples are given of those practical uses which are not too technical, and the limits of difficulty for most items are stated in the outline which follows. It will be seen that, in general, these limits are expressed in terms of competencies to perform certain tasks. This is one reason why the resulting outline of mathematical needs cannot be considered a course of study.

In selecting from the items on the check list those which should be included in the content of essential mathematics, attention was given first of all to frequency of use. On this basis a number of the items had obviously to be rejected. They occurred so seldom that, it was felt, instruction on them might properly be left to the Army branches concerned.

But it soon became apparent that frequency of use alone was not a sufficient basis for rejecting or including an item. In many cases, an item which was indicated as rarely used might still be of fundamental importance and necessary. For example, an inductee assigned to the Engineer Corps might not, for a considerable period, have to aid in constructing a bridge. But when he did, he should know how to construct, on the shore, a perpendicular to the line of the bridge. Again a supply clerk may spend most of his time issuing supplies. But at regular intervals he must do paper work that may involve adding rather long columns of figures, multiplying and dividing with decimals, and using fractions and per cents. While this part of his work is done only rarely or occasionally, he cannot be a good supply clerk unless he is proficient in these uses of mathematics. In cases of this kind, items were retained because of their crucial value in spite of the relative infrequency of their use.

Outline of Essential Mathematics for Minimum Army Needs

A. Reading and Writing Arithmetical Symbols

1. Whole numbers (to six digits).
   Serial numbers on identification tags and rifles contain as many as eight digits.
   Used in reading stock lists, equipment numbers, speedometer mileage, radio frequencies, blueprints.
2. Common fractions (denominators of powers of 2 through 64; 3, 5, 6, 10, and 12).
   Fractions with denominators in the ten-thousands are frequently used in map reading.
   Used in indicating parts of an hour, a mile; sizes of wrenches, bolts, nuts, drills; reading the carpenter's square.
3. Decimal fractions (to three places).
   In machine work it is sometimes necessary to read micrometer calipers to ten-thousandths.
   Used in indicating specific gravity of batteries; setting up correct frequencies in kilocycles; expressing caliber of projectiles; measuring spark plug gaps, valve clearances.
4. Per cents (to 100%).
   In certain instances per cents less than 1% are used to indicate composition of metals and the proportions of chemicals in solutions.
   The fractional equivalents of the dozen or so commoner per cents should be memorized.
   Used in indicating per cent of men assigned to details, apportionment of pay, proportions of ingredients in mixtures such as insecticides.

B. Counting

5. Counting by 1's, 2's, 5's, and 10's (to 500).
   Used in counting off in the squad and in infantry drill; counting automotive units; pacing for determining distances; cranking heavy tank engines; counting number of men for mess.

C. Operations with Whole Numbers

6. Addition (columns of not more than five addends of three digits each, and shorter columns of not more than six digits).
   Stock, mess, and supply clerks need to add much longer columns and to add short columns of numbers of more than six digits.
   Used in determining daily mileage for reports, amount of ammunition and materials consumed, total circuit resistance, total number of men.
7. Subtraction (numbers of not more than six digits).
   Used in reading map grids; determining directions; computing distances; determining the age of parachutes and pneumatic equipment in order to check their safety.
8. Multiplication (numbers of four digits by numbers of two digits).
   Calculation of total amount of rations, ammunition, and company supplies involves larger numbers.
   Used in finding distances when time and rate are given; keeping personal budgets; figuring costs; substituting in orientation formulas.
9. Division (numbers of four digits divided by numbers of two digits).
   Division of numbers of six digits is used in dealing with radio frequencies; reduction of map scales; estimation of quantities of materials in concrete construction.
   Used in determining direction and ele-

vation of sight; rationing of supplies and ammunition; changing units of measure as from inches to feet.

D. Operations with Common Fractions (Denominators limited as in 2 above)

10. Addition.
Used in determining over-all dimensions from blueprints; computing total demolition charges; calculating seam allowances.
11. Subtraction.
Used in blueprint reading; finding tap and die sizes; calculating depth of cut in machine work.
12. Multiplication.
Used in determining amounts of ingredients in mixtures; finding weight of castings; calculating total weight of rivets for repairs.
13. Division.
Used in solving electrical problems involving Ohm's Law; calculating tubing lay-outs; making switchboard installations.

E. Operations with Decimal Fractions (Denominators limited as in 3 above)

14. Addition.
Used in finding total money value of rations (to thousandths of a cent); adding currents in legs of circuits.
15. Subtraction.
Used in determining mileage from speedometer readings; checking aircraft specifications; machine shop work.
16. Multiplication.
Used in finding correct frequencies in electrical work; windage determination.
17. Division.
Used in making switchboard adjustments; determining costs; laying out rivet patterns.

F. Part-Whole Relationships, with common fractions, decimal fractions, and per cents

18. Finding part of a quantity.
Used in making pay deductions; determining amounts to be mixed in insecticides, in cooking, in mixing concrete; computing dial settings; finding part of supplies to be drawn; determining number of parachutes to be kept packed in accordance with a fixed percentage.
19. Finding what part one number is of another.
Used in finding speeds when different-sized pulleys or gears are used, outputs of transformers; determining slopes in setting grade stakes, the portion of a company that qualify in marksmanship.
20. Finding a number, given a part and its relative size.
Used in calculating amount of product in Army baking operations.

G. Ratio and Proportion

21. Understanding basic idea.
Used in statements of concrete mixtures, relative speeds and feeds, fuel mixtures.
22. Solving problems.
Used in finding scales in aerial photographs; calculating speeds when different sizes of pulleys or gears are used; figuring slopes.

H. Powers and Roots

23. Finding powers (squares and cubes).
Used in solving formulas.
24. Finding squares and square roots from tables.
Used in calculating sides of right triangles and diagonals of rectangles.

I. Graphs and Maps

25. Understanding grids and scales.
Knowledge of the fact that the numerator of a representative fraction (R.F.) is a distance on the map and that the denominator is the distance on the ground is necessary in map reading.
Such scales as 1/20,000, 1/62,500, and 1 inch = 1 mile are commonly found on military maps.
26. Determining directions from a map.
Used in determining direction of travel to reach an objective; orienting one's own position; determining direction in heavy gun fire.
27. Interpreting maps and graphs.
Used in discovering the location of artillery, the command post, hostile forces; determining the height of points on a terrain (from a map showing contour lines); scouting and reconnaissance; radio and signal communication.
28. Making graphs and maps.
The enlisted man is not commonly required to make maps and graphs, ex-

cept for rough sketches, but experience in making them is essential to proper understanding.

J. Tables

29. Reading from tables.
Reading to the nearest tabulated figure is all that is usually required.
Used in finding the angle of elevation for an artillery piece from firing tables; making corrections for windage; determining type of ammunition; finding strength of materials, fractional equivalents; converting measurements.

K. Formulas and Equations

30. Understanding of simple symbolism of algebra, such as a, K, a^2, b^3, $\sqrt{a}$, a_2, π.
Used in practically all branches of the armed services.
31. Simple formulas to be memorized: area of circle, triangle, rectangle; volume of cylinder, rectangular solid; distance, rate, and time.
Less used are the formulas for area of a parallelogram and of a trapezoid, and for conversion of temperature readings between Fahrenheit and Centigrade.
32. Substitution in simple formulas.
Used in such formulas as those listed in 31 and others as:
$N=\frac{3}{4}A$ (demolition of a steel beam);

$\frac{1}{R_T}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}$ (total resistance of three conductors in parallel).
Besides these simple formulas others of a greater degree of complexity are widely used in some branches of the service. For example, the demolition formula:
$N=R^3KC$ plus a per cent of N depending upon the size of N.
33. Solution of simple equations, such as:

$$3n=6 \qquad 2n-3=7$$
$$n+3=6 \qquad 2n+3=7$$
$$n-3=6$$
$$\frac{n}{3}=6$$

These equations usually result from substitutions in formulas.

L. Positive and Negative Numbers

34. Symbolism and meaning.
Used in indicating direction of flow of both alternating and direct current; machine gun firing tables; angle of sight instruments; calculating data for range cards on machine guns; determining direction; designating storage battery terminals.

M. Measurement, including understanding of basic units

35. Length, weight, area, and volume.
In many branches there is necessity for performing the fundamental operations with denominate numbers.
Used in determining area of fire, storage tank capacity, firing tensions, load limits; placing poles in field wire communications.
36. Temperature (C and F), angles (degrees, minutes, and seconds), time (24-hour clock).
Temperature measurement is used in determining tension for a wire, condition of engines, condition of patients; angles are used in machine shop work, machine gun and mortar firing, map reading; determining the number of degrees of revolution of a rotating shaft. The Army uses the 24-hour clock.
37. Metric system with simple equivalents.
Used in medical measurements; radio repair; designating gun caliber, dispensary supplies.
38. Measuring instruments.
Compass is used in determining directions; tape and rule are used in measuring lengths; protractor and compass are used in map reading; transit is used in road building and communications construction; calipers and dividers are used in the machine shop; steel square and level are used in carpentry; scales are used in measurement of weight.
39. Limits of accuracy or tolerances.
These range from the precise measurements in the machine shop, through the tolerances found in adjusting weapons, to the fairly rough estimates of road overhead clearances.
Used in measuring spark plug and breaker point gaps.
40. Estimation.
Load that a bridge will bear, weight of a load on a truck, number of men in a group, height of aircraft, distance in scouting, range for firing, distance marched, speed of vehicles.

N. Geometric Concepts

41. Point; straight, curved, horizontal, vertical, oblique, parallel, and perpendicular lines; angle; and slope.
Used in technique of fire; simple mechanical drawing and blueprint reading; map reading; setting the grade of a road.
42. Triangle (right, scalene, isosceles, and equilateral), parallelogram (square and rectangle), trapezoid, circle, ellipse, and regular polygon; prism, cylinder, cone, and sphere.
Used in cone of fire; aiming circle; machine cams; cylindrical drums; optical instruments; construction.
43. Similarity.
Used in relationships among automotive parts; pattern making; recognition of airplanes in flight.

O. Drawing and Construction

44. Use of ruler graduated in 32ds and in 10ths of an inch, and in millimeters; of compasses to construct circles; of protractors to measure and to draw angles.
Used in preparation of parts for welding; radio repair; pattern making; laying out range charts; determining azimuths.
45. Construction of a perpendicular to a line.
Used in construction of bridges; machine shop work; the thrust-line system of establishing a rendezvous; blueprint reading.
46. Understanding of views of a simple object as given in a scale drawing, blueprint, or sketch.
Used in fabrication of materials; motor mechanics; functioning of rifle; troop tactics; blueprint reading.
47. Knowledge of the 3-4-5 relation of the sides of a right triangle.
Used in carpentry; sighting mortars.

P. Miscellaneous

48. Averages: mean, median, and mode.
Used in calculation of average rations, average marksmanship scores; constructing a pace scale; Medical Corps statistics.
49. Rounding numbers.
Used in a very great variety of calculations.

General Suggestions with Respect to Instruction

Point of View

The Army needs enlisted men who have practical abilities in mathematics. So do the affairs of civil life. Moreover, the levels of mathematical ability which meet *minimum* Army needs for its enlisted men are relatively simple—essentially elementary school arithmetic and the less complicated concepts and skills of algebra and geometry. These facts emerged time and time again from interviews with training officers and from the check lists. If these facts have been stated more than once before in this report, they are repeated because they need to be repeated.

The public schools have not always developed the kinds of practical mathematical ability which are desirable both in the Army and in civil life. Instead, the mathematics period has too often been devoted to the development of skill in abstract computation. As a result the Army finds that many enlisted men who have had considerable work in mathematics are virtually helpless in practical quantitative situations. For example, men who in the geometry class may have skillfully used the Pythagorean Theorem fail to see its applicability and value in erecting the corner studs of barracks.

Almost all inductees will have had several years of school arithmetic. Hence, the needed course in mathematics will be in part remedial in character. But this fact is by no means an unmixed blessing, for among the consequences of traditional courses may be attitudes of indifference and even of hostility, misunderstandings, faulty habits, and ineffective procedures, all of which must be overcome before material progress of a positive kind can be undertaken. Furthermore, the instructional task is made more, rather than less, exacting by reason of the fact that the content must be *useful*. The truth of the statement and its implications for instruction may not be evident at first glance.

Points of Emphasis

Remedial instruction.

Responsibility for remedial instruction in arithmetic lies with the teacher of secondary mathematics. The same responsibility, as a matter of fact, has always lain with the teacher of secondary mathematics, though he has not been inclined to accept it. He may have acted on the assumption that no such remedial instruction is necessary; or on the assumption that, if it is necessary, it should be someone else's job; or on the assumption that if his students, with or without arithmetical competence, can be successful in the sequential mathematics courses, his obligation is met. The present emergency invalidates all these assumptions, as indeed they would long since have been invalidated if we had but evaluated sensibly the consequences of arithmetical illiteracy in personal and social terms. Be that as it may, intelligent mastery of arithmetic is now recognized for its importance, and to the extent that it is not in the possession of pre-inductees, they must be aided to acquire it.

The usual program of remedial instruction, which is merely to assign more practice of the previously ineffective types, must be abandoned; it does not meet the need. The teacher of secondary mathematics must be genuinely anxious to diagnose arithmetical deficiency (and mathematical deficiency in general) as the first step. If the student cannot correctly add columns of two- and three-digit numbers, the teacher needs to know why. Perhaps the student has never learned to add, relying upon simple counting to satisfy his needs; or, perhaps he is deficient in addition merely because he does not know the addition facts as well as he should. If the student adds $\frac{1}{2}$ and $\frac{1}{4}$ and gets $\frac{2}{6}$, again the teacher needs to know why before undertaking remedial instruction. Perhaps the student fails utterly to understand fractional values; or, perhaps he understands such values but does not use them in computation, substituting purely mechanical procedures. It should be obvious that the particular form which remedial treatment will take is directly dependent upon the type of deficiency which is to be corrected.

New Concepts.

But the course in essential mathematics should include more than remedial work, whether arithmetical or otherwise. It should include the development of new concepts (or concepts so incompletely developed in the past that the adjective "new" is warranted) and the provision of many and varied experiences in applying what is learned.

Procedures for teaching new concepts may be illustrated by using here a single example, namely, the development of concepts of units of number. This one illustration will serve because the course of learning concepts in this area is truly typical of concept development in mathematical areas generally.

The learner's beginnings with units of number are concrete. The smaller numbers first take on meaning because they are represented by groups of objects which can be readily counted and otherwise manipulated. Continued experience, particularly if that experience be with different objects, different groupings, and for different purposes, eventually frees the concepts of the smaller numbers from their dependence upon particular details and makes the freed concepts available for relatively abstract thinking.

But direct sense experience has its limitations when it comes to learning the meaning of larger numbers. We commonly count to determine the precise number of perhaps twenty-five or fewer objects, but rarely do so to determine a number of a hundred or a thousand objects, to say nothing of ten thousand, a hundred thousand, or a million. To have any understanding of these large numbers we must utilize something other than direct sense experience.

That "something" is available in the smaller groups of number which have already been acquired, such as groups of 2, 5, and 10. Knowing that 25 is five 5's, that 42 is six 7's, and that 72 is eight 9's, we can understand 25, 42, and 72 without the necessity always of assembling piles of objects and laboriously counting to find totals. The smaller groups thus serve to carry us beyond the limits of direct experience, but they could not do so, were they not themselves based upon direct experience to give them meaning.

Since our number system is decimal in character, the multiples of 10 naturally enter into an understanding of larger numbers with values far beyond those sensed by the average person. For example, the number 8543 may be thought of as 85 hundreds, 4 tens, and 3 ones; and this reconstruction of the number makes it sensible to whoever really understands a hundred even though he cannot comprehend a thousand. Or, the same number may be thought of as 8 thousands (with 10 hundreds in each thousand), plus 5 hundreds, and 43. And as the individual gains further number knowledge, even larger numbers take on new meanings. For example, he may think of 751,823 as about $\frac{3}{4}$ of a million; of a million, as about 1000 thousand. None of us has experienced a million directly and concretely; but all of us, by making use of multiples of smaller units, understand something of the meaning of a million, an abstraction which is far removed from direct sense experience.

The foregoing paragraphs on units of numbers then, illustrate the complexity of the instructional task involved in developing mathematical sense,—they illustrate its complexity and at the same time suggest the general pattern which instruction must take in order to achieve the goal set. The movement is from concrete (where the meaning is most easily apprehended) to the abstract (where the idea or skill is freed of its particularized content) and back again to the concrete (where the freed idea and skill can be applied usefully). To the extent possible within limited space this movement will be apparent in the sections which follow, where suggestions are offered for teaching the minimum mathematical topics.

Applications.

In this section two special responsibilities in the teaching of essential mathematics have been discussed: (1) the teacher must provide for remedial instruction based upon real diagnosis of shortcomings in previously taught skills, and (2) he must undertake to develop new mathematical concepts. He has still another task: (3) he must provide ample experiences in application. From the standpoint of learning, this third task completes the cycle which has just been mentioned, from concrete to abstract and back again to concrete. "Back again to concrete" means use in practical situations. It is not uncommon to use concrete visual and other sensory aids to derive the formula for the area of a circle by dividing circles into equal sectors. This procedure, which is wholly admirable, serves to develop the abstract idea from concrete experience. It is less common in instruction, however, to take the next step and have students apply their abstract knowledge to new concrete (practical) situations. In the example here under discussion, it is less common to have students actually determine the area of the base of a cylinder, preliminary to determining its volume, by measuring a cylinder directly to ascertain the elements for the mathematical formula. Yet, without this experience in application, knowledge of the formula cannot guarantee ability to *use* the formula as it is used in Army and civilian life.

Again, it is not uncommon to develop the concept of a mile by stating that it is a distance equivalent to 5280 feet (feet being objects of direct sensory experience) and to 1760 yards (yards also being experienced rather directly). Yet the term mile can have but little meaning if taught only in this way; after all, 5280 feet are a great many feet, and 1760 yards are a great many yards. Somewhere in the teaching procedure the abstract concept of mile needs to be checked against direct experience: the student needs to pace a mile, timing himself, and counting the number of paces required. He needs also to identify the distance of a mile as that between such-and-such familiar objects. In the absence of such experiences the concept mile can scarcely function valuably in practical situations.

The purpose of the preceding several paragraphs has been to suggest ways of developing *mathematical sense.* Mathematical sense is what the Army requires, and it is what civilian life also requires for more effective adjustment to the quantitative aspects of our culture. More and more, methods of instruction must tend toward laboratory procedures. Such teaching procedures provide the direct first-hand experiences that will insure that concepts and skills will function in practical situations.

Importance of meaning.

Teachers who are not accustomed to thinking about mathematical instruction in terms of meanings may raise one or more of several questions as they continue through the report. They may ask: (1) "But I thought what is wanted is a practical course, and everywhere I encounter theory. How can my course be practical if I teach all this theory?", or (2) "Where are the proposed meanings to be taught—in the lower grades, or in the junior and senior high school, or at both levels?", or (3) "Granted that these meanings are important, can they be taught even in high school to students who are only average or sub-average in ability?", or (4) "Must I try to teach these meanings even if it means neglect of some degree of computational skill, or should I insure computational skill first, giving such time as remains to meanings?"

These questions are all pertinent, and they deserve answers even though the answers must be brief and somewhat dogmatic. (1) One does not have to become theoretical when one discusses meanings; meanings are really practical. It has long been pointed out that skills cannot be used intelligently if they are not learned intelligently. Meanings function practically in intelligent use. We can prepare the student for particular jobs without bothering much about meanings; but the student so prepared is helpless when the job is changed. It is meanings that are transferred, and it is this transfer that is essential if the student is to be adequately equipped to face new situations. (2) The proposed meanings should be taught wherever needed for intelligent learning. They should certainly be taught in the lower grades when concepts and skills are first introduced, and experience has shown that they *can* be taught at this time. But if these meanings have not been acquired in the lower grades, they become an essential part of mathematics courses in the junior and senior high school. (3) It has not yet been demonstrated that average and somewhat sub-average students cannot acquire the meanings discussed in this report. Certainly past experience in this respect is an unsafe guide, for in the past, effort expended in this direction has not always been continuous and expert. Still, realism requires us to face the fact that some students may be incapable of mastering the proposed meanings. In such cases, and they should be relatively few, it is probably best to concentrate on computational skill. (4) Nothing in this report should be interpreted as suggesting lack of concern for computational skill. If the report seems to over-stress meanings and understanding, it is only because these mathematical objectives have been too long understressed.

Extent of Treatment

To treat adequately all the changes in instruction which would follow from what has been said would require a textbook, or even several textbooks. No such amount of space is available in this report. It is possible here only to sample the suggestions which might well be offered for the improvement of instruction. Yet, the necessarily brief and incomplete discussions of the selected suggestions are offered with the firm conviction that, if heeded, the recommendations with regard to teaching procedures will move instruction considerably closer to the goal. The careful reader of this report will scarcely be able to miss the general tenor and purpose of the suggestions, and the alert teacher will catch the implications for instruction which cannot be treated here.

Sources of Instructional Material

Those responsible for the establishment of courses in mathematics which will cover the minimum essentials outlined in the preceding section will be concerned with the possibility of securing suitable instructional material. It will be found that many of the textbooks in general mathematics made for use in the junior and senior high school will approximately cover the essentials listed previously. To be satisfactory for the intended course the text should have a treatment of arithmetic (operations with whole numbers, fractions, decimals, and per cents). It should include graphs and scale drawing leading to work with maps and blue print reading; literal numbers leading to simple formulas and equations; signed numbers; geometric concepts, drawings, and constructions; keeping simple accounts; and measurement. It should emphasize the practical applications of mathematics. It may also contain the uses of the sine, cosine, and tangent, and possibly measures of central tendency. When the available text does not include all of the essentials outlined above, supplementary material can be provided by the teacher. In any case, whatever text is used, the teacher should see that the problem material is made real and practical. For this reason the teacher should collect such material.

Specific Suggestions with Respect to Instruction

A. Reading and Writing Arithmetical Symbols (Items 1–4)

This topic may seem to be rather elementary for pre-inductees. It would be, if only the ability to read and write abstract symbols without knowledge of their logical structure and interrelations were the goal. The purpose, however, in teaching students to read and write whole numbers, common and decimal fractions, and per cents is to raise their understanding of these symbols to a more functional level. Instruction is needed which shows the logical basis of such symbols and those relations which make for rich number concepts.

Whole numbers.

Number denotes quantity by a measuring off into so many units. Although 10 and powers of 10 constitute the basic units upon which number is built, they are not the only units which enter into the comprehension of number. Let the reader test himself by reading the following numbers: 144, 325, and 81. As he reads these numbers, units of 12, of 5 or 25, and of 9, and possibly others in addition to units of 10 and powers of 10 occur to him. Probably no two individuals would measure all the amounts mentioned above by the same set of basic units. However, when numbers of two or more digits are read, units of 10 and powers of 10 play an increasingly important part. A number such as 43,642 may suggest 40,000 which in turn is subject to further interpretation. Many opportunities present themselves in traditional courses in mathematics to enrich thus the student's concept of number.

Throughout this report emphasis is placed upon the importance of place value (inherent in our number system) in the comprehension and manipulation of numbers. Traditionally, place value has been taught by rote methods, and students memorize merely the names of place values. Charts showing place values appear in most texts. Without accompanying instruction to make place value meaningful, these charts make little contribution to understanding. Since place value depends upon powers of 10, the need for extensive experience with these units is obvious, experience which reveals decimal relationships in ascending and descending orders, so that 1000 is known as 10 hundreds, 10,000 as 100 hundreds, 1 million as 1000 thousands, etc., and similarly, so that 100 is known as 1/10 of 1000, 1000 as 1/1000 of a million, 10 as 1/100 of 1000, and the like.

Common fractions.

Students' difficulties with common fractions stem from two sources: failure to understand the ideas which fractions express and failure to see these meanings in the form of fractional notation. For example, students may not understand $\frac{3}{4}$ as an expression of ratio because they are ignorant of the concept of ratio or because they are unable to see that $\frac{3}{4}$ *is* an expression of a ratio.

We use fractions to show: (1) a part or parts of a whole or of a group, (2) indicated division, and (3) ratio. There are differences enough among these meanings and uses to demand the attention of the teacher.

(1) The phrase "$\frac{3}{4}$ of an orange" means that a single orange has been divided into four equal parts and that three of these parts are under consideration. The denominator, 4, tells the relative size of the parts; the numerator, 3, the number of parts being considered.

Fractions as applied to single objects occasion little trouble for students, but the corresponding application to groups of objects ($\frac{3}{4}$ of 12 oranges) and to abstract numbers ($\frac{3}{4}$ of 12) is not so readily grasped. Yet the extension does not seem to be very great. Students who fail to make the extension have in all probability been hurried and have responded in the only way that is open to them, by superficial learning.

The nature of appropriate instructional procedures should be clear by now to the reader of this report: Start with concrete experiences, and continue at this level as long as may be necessary to engender the needed meanings. Let the student use real or representative objects to find answers for as many examples as necessary of the type: $\frac{2}{5}$ of 10 books, $\frac{3}{8}$ of 16 sheets of paper, $\frac{3}{4}$ of 12 erasers, $\frac{2}{7}$ of 21 pencils. At this stage of learning, answers should be found by placing objects in groups or by indicating the division by drawing lines around representative objects. The example may then be written in abstract form, together with the answer, and the functions of denominator and numerator made clear by referring both to the fraction and to the operation completed with objects. The transition to purely abstract numbers and computation comes later. At first it is wise to check abstract computation by returning to concrete situations. When real understanding has been achieved, practice needs to be assigned to promote proficiency and to fix the new computational habits.[5]

(2) The fraction $\frac{3}{4}$ may be considered an indication of a division. Here the major instructional task is to teach the significance of the form of notation: the line between numerator and denominator means a division. In Army usage this meaning of fractions occurs chiefly in formulas and in finding per cents for fractions or fractions for per cents. These uses are common enough and important enough to require that this meaning of fractions be thoroughly understood.

[5] When these meanings are first taught in the lower grades, many class hours will be needed. When they are reviewed or retaught in later grades, much less time may be necessary. In the senior high school it is possible that a few minutes will suffice. However, even in the senior high school the teacher should begin in this simple concrete way and should not pass to the abstract phase until it is clear that the concrete phase has been mastered. *The caution in the last sentence applies generally to all aspects of instruction in essential mathematics.*

(3) The fraction $\frac{3}{4}$ may be considered the ratio three out of four. This meaning is important enough in the Army and in civilian life to require careful instruction. The first task for the teacher is to develop the idea of ratio, and this topic is discussed later in Section G. The next task is to relate the meaning of ratio to the fractional form of notation. Difficulties are more likely to arise in teaching the first than in teaching the second aspect of this use of fractions.

Part of the problem of teaching the meaning of fractions is to establish the relationships between equivalent fractions, relationships which are essential if later on the student is to reduce fractions and to find common denominators with any degree of intelligence. Again, the start should be made with concrete materials, in order that the relationships may be readily apparent. With such materials the student should experience little difficulty in seeing that the following series (and others like them) are true:

(1) $1=\frac{2}{2}=\frac{4}{4}=\frac{8}{8}=\frac{16}{16}$

(2) $\frac{2}{3}=\frac{4}{6}=\frac{8}{12}=\frac{16}{24}$

(3) $1=\frac{2}{2}=\frac{3}{3}=\frac{4}{4}=\frac{5}{5}$

(4) $\frac{1}{2}=\frac{2}{4}=\frac{3}{6}=\frac{4}{8}=\frac{5}{10}$

If such relationships are taught without reference to concrete objects, the price may be knowledge which will not function when needed.

Decimal fractions.

Decimal fractions are related to common fractions for their meaning and to whole numbers for their notation. Failure to grasp either of these basic relationships or the reciprocal relationship between them leads to misunderstanding of decimal fractions and to inability to deal with them in a sensible and effective manner.

It is common practice to have the student derive decimal fractions directly from equivalent common fractions without preliminary experience in measuring with decimal units; units of 10 to start with, then units of 100. It is suggested that instruction on the meaning of decimal fractions start with such measuring experiences. First, the student may be supplied with strips of paper divided into ten equal squares. Parts of the whole strip are then expressed in terms of tenths of the whole, written as common fractions, then as decimal fractions. Or, mimeographed series of lines divided into ten equal segments can be furnished to the student with directions to draw heavy vertical lines to show 3/10, 9/10, 4/10, etc., the equivalent decimal notations then being entered opposite the corresponding drawings. Later on, squares of graph paper with ten equal squares to the side may be used in a similar way to develop the meaning of hundredths. Parts of the whole square can be blackened to show 7/100, can be dotted to show 21/100, can be crosshatched to show 37/100, and so on, the common fractions then being translated into decimal equivalents.

Once experiences in direct measurement by means of decimal units have contributed their share of meaning to the understanding of decimal fractions, it is time enough to introduce the general relation between decimal fractions and equivalent common fractions, e.g., $\frac{3}{4}$ and .75, $\frac{2}{3}$ and $.66\frac{2}{3}$, etc. These relationships will hardly be understood if the learner merely memorizes the common fractional and decimal equivalents. First, the student needs to know how to translate common fractions into decimals, as well as the reverse process, beginning with the more common decimal and fractional equivalents and continuing with less common pairs, e.g., $\frac{5}{9}=.55\frac{5}{9}$, $\frac{3}{7}=.42\frac{6}{7}$, etc. Supported by this understanding which will enable him to find the correct fraction or decimal if he forgets it, the student may then safely be encouraged to memorize a dozen or more of the commoner pairs of equivalents.

It is frequently said that the principles governing the writing of decimal fractions are the same as those governing the writing of whole numbers, and therefore that learning the former skill requires merely the extension of what is already known about whole numbers. These statements are true, but they are likely to oversimplify the learning situation and to blind teachers by false inferences. In the first place, this statement rests upon the assumption that students, when they come to decimal fractions, *have* something to extend. This assumption may or may not be valid. It is possible to read and write whole numbers with but the slightest grasp of the principles of place value. In the second place, the extension so hopefully expected does not seem to be made as easily as is anticipated. Even children who understand place value in whole numbers seem to have plenty to learn about reading, writing, and interpreting decimal fractions,—else why their errors? The truth is that this extension *is* a simple matter to the person who has made it, but it is not so simple to the one who is making the extension.

The foregoing statements do not imply that it is impossible to teach the meanings of decimal fractions or to utilize knowledge of place value previously acquired. The purpose of the statement is, rather, to emphasize the fact that there *is* a teaching task in connection with the meanings of decimal fractions and the abilities to read and write them, a task which is ignored or minimized only to the detriment of the student. It is not enough that the student be able correctly to read and write .4, .04, and .004. To be protected from absurdities of interpretation the student must understand the relationships as well as the differences among these expressions. As has already been suggested, these relationships rest in part upon sound concepts of fractions and in part upon a functional grasp of place value. The decimal fraction .4 cannot have the same value as .04 because .4 means 4/10 while .04 means 4/100; or, the 4 of .4 is in tenths' place and means 4 tenths and the 4 of .04 is in hundredths' place and means 4 hundredths. Yet, the two expressions at the same time are alike in the sense that the 4 in both decimal fractions is a numerator. In 4/10 and in 4/100 the denominator is apparent; in .4 and .04 the denominator is not apparent but implied. Difficulties in reading, writing, and interpreting decimal fractions will certainly arise if the student does not understand that the *places* of the significant figures in decimals give him the implied denominator.

One further word about the meanings of decimal fractions. The student needs to understand such relationships as: (1) for abstract numbers, .4 = .40, or .400; (2) for measurements, .496 and .516 are approximately, although not exactly, equal to .5 (5/10), or to .50 (50/100), or to .500 (500/1000); and (3) as relative magnitude, .4 is more than .39 and less than .402. These relationships are vital; they are found commonly in Army and civilian jobs. Difficulty of interpretation is likely to result from the fact that the equal and unequal decimals contain varying numbers of digits. Instruction on the meaning of decimal fractions must include attention to these and similar relationships.

Per cents.

The phrase, "a 2% slope," should mean to the soldier a slight rise in the terrain. In such an application of per cents, which is common and important both in Army and in civilian life, we come face to face again with the problem of meaning. Moreover, when per cents occur as parts of more complex situations which require computation (see Section F), the meaning of per cents is even more crucial.

Before the student studies percentage, he is usually familiar with such expressions as 3 out of 4, 9 out of 10, 45 out of 50, and the like. Moreover, having studied both common and decimal fractions, he knows that these relationships can be expressed as $\frac{3}{4}$ or .75, 9/10 or .9, and 45/50 or .90, respectively. In these expressions he has used various bases: 4, 10 and 50. The problem in teaching percentage is to show the student that *all* bases can be standardized to 100, so that the parts are known as hundredths or per cents.

The last statements imply more than mechanical facility in dividing numerators by denominators in the case of common fractions and pointing off the answers to two decimal places, and in substituting the sign %, written after the number, for the decimal point written before the number in .50 and .73. Mechanical habits alone hardly suffice for translating into per cents such common fractions as 11/3 and 1/27, or such decimal fractions as .1 and .507, to say nothing of 4.007.

The base of hundredths needs to be meaningful in its own right. The preceding section (pages 256–257) contains suggestions for the use of graph paper to teach the meaning of two-place decimals, and the same materials may be used in the case of per cents. To teach the meaning of per cents from 1% to 100%, a strip of graph paper made up of ten equal squares, each divided into ten equal parts (rectangles) may be used. Each part is 1%, so that fifteen parts equal 15%, sixty-nine parts equal 69%, and so on. To teach the meaning of per cents smaller than 1%, a strip of graph paper ruled 10 squares to the inch and 10 inches long will provide a total of 1000 small squares. Each of the small squares is 1/1000 or .1% of the whole, and practice can be given in designating such per cents as .9%, .5%, etc.

Meanings of per cents larger than 100% are also important in the Army and in civilian life, and should therefore be taught. In the Army the chief use of such large per cents is restricted for the most part to expressions of ratio. In the expression 6 is 150% of 4, the relationship, not at first apparent, becomes apparent if 6 is first thought of as 1.5 times 4. In civilian life the problem of buying for $4 and selling for $12 involves the relationship of 300% if selling price and cost are compared (12/4, or 3/1), and of 200% if gross profit and cost are compared (8/4, or 2/1).

B. Counting
(Item 5)

It might well be assumed that prospective inductees will know how to count. It does not necessarily follow that they can use counting efficiently in practical situations. This functional use of counting implies the abilities: (1) to place with accuracy a group of objects in a one-to-one correspondence with the natural numbers; (2) to count by units of such groups as 2's, 4's, 5's, and 10's; and (3) to recognize identical units and find the total by multiplication.

The ability to determine the one-to-one correspondence between a group of objects and the natural numbers demands something more than rote counting. It implies that thorough familiarity with numbers which has been previously noted as mathematical sense together with such an orderly procedure as tallying (see Section P, page 280).

The ability to count in such multiple units as 2, 4, 5, or 10 is not only useful in many Army and civilian tasks but also serves to provide enriched concepts of number and a more thorough appreciation of factors, of multiples, and of products.

Instead of counting by other groups such as 3, 6, 7, or 9, a sensible recognition of number will direct the student to count the number of such groups and use multiplication. Thus if a group of soldiers march aboard a transport in ranks of 3, the counter would determine the number of ranks of 3 and find the total by multiplication.

In any case the ability to count with assurance and skill implies a familiarity with number, with units of multiples of number and with tallying. Only thus can the practical aims of minimum Army needs be met. This does not mean that there is no value in having the student count by units of 3, 6, 7, or 9. Counting by such units provides excellent practice in adding by endings, and in proceeding from decade to decade concepts of number are more enriched than when the individual counts by 1's.

C. Operations with Whole Numbers (Items 6–9)

Computation.

Our methods of computation and the forms (algorisms) we use are based directly upon the fact that ours is a decimal system, with 10 as the base. Consider the following computations:

A.	B.	C.	D.	E.
49	72	56	39	32
+36	−37	×7	×24	8)256
85	35	392	156	24
			78	16
			936	16

In A, when we add 9 and 6 in the ones' column, we secure the total of 15 (ones), which we then break up into one ten and 5 ones. The 5 ones are represented in the sum by the 5, which is written in ones' place. The one ten is carried to the tens' column where it is combined with 4 (tens) and 3 (tens) to give 8 (tens). We do not carry "1"; instead, we carry "1 ten," and we then write the total of 8 under the 4 and 3 because we must: the 8 stands for 8 tens.

In B, we cannot subtract 7 (ones) from 2 (ones), so we borrow ten (not "1") from the 7 (tens), leaving 6 (tens). The subtraction continues, 7 from 12, 3 from 6; and the digits of the remainder are written where they must be written—5 in the ones' place, 3 in the tens' place. These positions are dictated by the fact that they stand for ones and tens, respectively.

In C, the first multiplication (7×6) yields 42, which may be thought of as 4 tens and 2 ones. The 2 for the ones is written under the 6 and the 7, because they, and the 2, must be in the ones' column. The 4 (tens, not "4") is carried, and is added to the next product which is not 35, but 35 tens. Hence, the digits in the sum 39 (tens) must be written with the 3 in hundreds' place and the 9 in tens' place.

In D, the second partial product (78) is really 78 tens, for it results from multiplying 39 by 2 tens (the 2 of 24). For this reason, and for no other, the 8 must be placed in the tens' column and the 7 in the hundreds' column.

In E, the first partial dividend is really 25 tens, and not 25. Hence, the quotient figure 3 must be written in the tens' place, above the 5 and the 4, which are already in the tens' column. And the second quotient figure, 2, can go nowhere else than above the 6 in the ones' place, because the second division is 16 (ones)÷8.

Such is the rationale of the five computations, a rationale which is derived directly from our decimal number system. This rationale can be understood only if one already understands the composition of whole numbers in terms of place value. Mechanical habits of computation and the rules that go with them serve so long as (1) they are remembered and (2) they are employed in completely similar situations. But intellectual habits which are not built upon understandings are notoriously easy to forget,—as witness the great losses in computation on the part of adults and high school students. Moreover, one rarely uses mathematics in situations which resemble closely those in which it was learned. To the student of mathematics who believes in meaningful instruction, it is not surprising that enlisted men are frequently incompetent in simple arithmetical computations with whole numbers, nor that they are even less competent when computing with common and decimal fractions and when using per cents.

The mathematics course for prospective inductees should not be restricted to drill in computation. Unenlightened drill will carry the student no further than the point which he has previously attained and will subject him again to immediate and drastic losses. Instead, teachers of mathematics should see that their students acquire the meanings and understandings which will protect them against losses and will make them capable of intelligent behavior in quantitative situations.

Approximation and estimation.

Understanding of the number system, of the functions of the fundamental operations, and of principles governing computation is of great assistance in estimation and approximation. In this connection, the student needs to know the effect of the operations on the whole numbers with which he deals. When whole numbers are concerned, in addition the answer is larger than is any addend; in subtraction, the answer is smaller than is the minuend; in multiplication, larger than either the multiplier or the multiplicand; in division, smaller than the number divided. Exceptions arise of course when 0 is added, subtracted, multiplied and divided.

Knowledge of these relationships allows one to detect at once the absurdity of such answers as: 66 for $89+23$; 63 for $38-25$; 340 for 5×468; 631 for $462\div 2$. Knowledge of these relationships, together with understanding of the number system and of computation, enables the student to tell without paper and pencil that the sum of 389 and 512 must be in the neighborhood of 900 (389 approximates 400; 512 approximates 500); that the difference between 849 and 763 must be nearer 100 than it is 350 (the example is approximately $850-750$); that the product of 8 and 78 must be about 640 (78 is nearly 80), and that the quotient when 596 is divided by 7 will be more than 75 and less than 100 (596 is about 600).

The ability to detect absurdities and to know what are reasonable answers for given number relationships is important in the Army. Evidence which supports this statement was continually met. Much of this evidence, at the same time, revealed the importance of mental arithmetic (calculation without pencil and paper).

Problem-solving.

When one performs abstract computations, one is always told precisely how the numbers are to be treated. For example, in the case of addition, one sees the addition sign or the word *sum* or *add*. In subtraction, one sees the subtraction sign or the word *remainder*, *difference*, or *subtract*.For multiplication and division there are comparable cues. In computation therefore the student, no matter how mechanical his habits, is not confused by the need to decide which process to use: the cues tell him.

Such, however, is not the case when the student deals with problems, whether the problems are encountered in school or outside. Consider the following verbal problems, stated for the sake of brevity in textbook form, which parallel the abstract examples discussed above:

A. If you count 49 pairs of shoes on one shelf and 36 pairs on another shelf, how may pairs did you count on the two shelves?

B. If only 37 men of a detail of 72 report on time, how many are late?

C. If it takes 7 men 56 hours each to complete a job, how many hours of work are required in all?

D. If socks sell for 39¢ a pair, for how much would 2 dozen pairs of of the same kind sell?

E-1. If you can pack 8 shells in a box, how many boxes will be needed for 256 shells?

E-2. If 8 details of the same size are to to be made up from 256 men, how many men will there be in each detail?

Where, in Problem A, is the addition sign, *add*, or *sum?* Where, in Problem B, is the subtraction sign, *difference*, or *subtract?* And where in Problem C to E-2 are the cues for multiplication and division? Obviously, they are not present. Instead, the student must have some way in which to determine for himself the process to be used in each problem.

It is precisely here that the student, however skilled in mechanical computation, tends to fail in problem-solving: tell him what process to use and he will (or may) secure the correct answer, but he cannot decide for himself the process which is required. The teacher in the intermediate grades recognizes the comparable situation when her pupil comes to her with the question, "Yes, but do I add, or subtract, or multiply, or divide?" It is most unfortunate that the significance of this question is not grasped. If it were, the child would not be *told* what process to use, but would be guided toward the correct decision in a way which would have permanent good effects.

The fundamental operations are simply mathematical methods of answering a few basic questions. (1) We *add*, when we must answer the question, "How many in all?", "What is the total?", or something similar, and when the numbers involved are not of the same size. (2) We *subtract*, when our question is, "What is the other part of the number?", or "What is the difference?" (3) We *multiply*, as we add, in response to the question, "How many in all?", or "What is the total?", when the numbers are of the same size. (4) And we *divide* to answer two questions: (a) "How many times does this larger number contain the smaller?" (for example, note Problem E-1 above), and (b) "How many are there in an equal part or share of a number when there are so many parts or shares in the whole?" (see Problem E-2 above).

True, the learning situation seems to be over-simplified when each process is said to answer one or two questions. Actually, the number of language forms which set the need for each process is very large; but all the different language forms for a given process can be reduced to one or two basic questions. The teaching task, therefore, is not to prescribe vocabulary drill on a few language forms nor to try to find all possible language forms for each process, but to supply experiences with enough different forms to develop familiarity with the generalized question or questions. Equipped therewith, the student is able to deal effectively and confidently with practically any need for each process, no matter in what words the need is couched.

Teaching the meanings of the operations.

The basic meanings of the fundamental operations are readily taught if simple, concrete, practical situations are used, preferably situations which may be represented with objects or by drawings or dramatization. When the learner actually *combines* two groups of 5 persons and 7 persons to get a total, by forming from them a single larger group, he sees at once (1) the nature of the process (combination of numbers of unequal size), (2) the kind of occasion which calls it forth, and (3) the effect of the operation on the original numbers, with the relation of the answer thereto. Likewise, when the learner actually makes up 6 equal sets of books from a total of 72, in response to the question, "How many sets of 6 books each can I get from 72 books?",—when he actually counts out the books (real books, or symbolic objects for them), he gains insight into the process of division.

Nothing is gained in teaching the meanings of the processes by using very large numbers, for these meanings are less apparent when the extra factor of large numbers is present. Nor should purely abstract situations be used at first. Later on, such abstract situations are warranted, but the number relationships in such situations should at first be presented as tangibly as is possible, in order to make the transition easy. It takes time for meanings to develop, time which must be filled with appropriate experiences.

D. Operations with Common Fractions[6] (Items 10–13)

Providing the student has real under-

[6] For the sake of brevity, mixed numbers as such are not included in this report. It is assumed that the teacher of mathematics will

standing of the various meanings of fractions, of reduction of fractions, and of equivalent fractions, he should readily learn how to compute with common fractions. Without these understandings, he can acquire only mechanical skills which are forgotten as quickly as they are "learned," and which in any circumstances are scarcely serviceable. On the other hand, equipped with these understandings and given adequate practice, he will need to learn little more than the idea of the common denominator in order to add and subtract (and possibly to divide) with fractions. Multiplication by and of common fractions raises few difficulties, save as indicated below; and a real understanding of this process will forestall confusion with other processes.

The concept of the common denominator is an extension of the ideas of reduction and of equivalent fractions. Indeed, within the range of denominators usually met in the Army, common denominators are virtually taken care of by these ideas. But at this point it is desirable to extend instruction beyond the minimum in two respects. (1) It is important to teach the meaning and function of the common denominator,—what it is, why it is needed, what it does for us that could not otherwise be done. (2) Occasionally Army inductees must deal with fractions not in the minimum list, under conditions which call for the common denominator. To prepare for such instances it will usually suffice to have students multiply the largest given denominator by 2, or by 3, and so on, until they have found the common denominator.[7] Finding the common denominator by the method of factoring need not be taught for minimum Army use, or for most uses in civilian life.

Once the common denominator has been found, addition and subtraction should go forward smoothly, answers always being expressed in terms of the common denominator or a reduced form thereof. Use of the common denominator can also be recommended for teaching division by and of a fraction. When a fraction is the divisor, the usual rule, "Invert the fraction and multiply," is taught immediately without rationalization. Thus, $\frac{4}{9} \div \frac{5}{6}$ becomes $\frac{4}{9} \times \frac{6}{5}$. If the common denominator method is used first, the example $\frac{4}{9} \div \frac{5}{6}$ becomes $\frac{8}{18} \div \frac{15}{18}$, and the answer is seen to be $\frac{8}{15}$, since the denominator 18 is common to both numerators, 8 and 15, and can therefore be disregarded. Later on, the inversion method can be taught to those who can profit by it for precisely what is, a short cut which yields the correct answer with a minimum of time.

Estimation and approximation.

With common fractions, as with whole numbers, it is not always necessary (and it is frequently impracticable) to use paper and pencil to find answers in Army situations. In such cases the individual must estimate and approximate, and he will be successful in so doing only to the extent (1) that he understands the meanings of fractions and (2) that he knows the effect of the fundamental operations on the fractional values with which he deals. The first has been treated in a separate section above; the second needs explanation.

When one adds whole numbers, the sum is larger than is any addend. This relationship holds also in the addition of common fractions. So does the corresponding principle for subtraction: the remainder or difference in subtraction is smaller than the quantity subtracted from, whether whole number or common fraction. But not so for the relationship between factors and answers in multiplication and division, where the situation with respect to whole numbers differs from that for common

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know how to apply the suggestions given to operations with such numbers. Similar considerations seem to justify the omission of any discussion of mixed decimals (4.7, 20.054) in the following section (Section E).

[7] This method will not be efficient in some Army jobs, as for example, indexing on a milling machine, where it is at times necessary to find the common denominator for fractions with such denominators as 19 and 21.

fractions. When two whole numbers are multiplied, the product is larger than is either number; when divided, the quotient is smaller than is the number divided. The differences in the case of computation with fractions can be made clear by analyzing relationships like those below.

When we multiply a whole number by a proper fraction (see A below) or a fraction

A. $4 \times 6 = 24$
$2 \times 6 = 12$
$1 \times 6 = 6$
$\frac{1}{2} \times 6 = 3$
$\frac{1}{4} \times 6 = 1\frac{1}{2}$
etc.

B. $\frac{1}{2} \times 1 = \frac{1}{2}$
$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$
$\frac{1}{2} \times \frac{1}{8} = \frac{1}{16}$
$\frac{1}{2} \times \frac{1}{16} = \frac{1}{32}$
etc.

C. $4 \times 6 = 24$
$4 \times 3 = 12$
$4 \times 1\frac{1}{2} = 6$
$4 \times \frac{3}{4} = 3$
$4 \times \frac{3}{8} = 1\frac{1}{2}$
etc.

by a proper fraction (see B), the product is smaller than the multiplicand. When we multiply a fraction by a whole number, on the other hand, as in C, the product is larger than the multiplicand, for obvious reasons.

When we divide a whole number by a proper fraction (D) or a fraction by another proper fraction (E), the quotient is larger than the dividend. When we divide a fraction by a number larger than 1, as in F, the quotient is smaller than is the dividend.

D.	E.	F.
$12 \div 4 = 3$	$\frac{1}{2} \div 1 = \frac{1}{2}$	$\frac{1}{2} \div 1 = \frac{1}{2}$
$12 \div 2 = 6$	$\frac{1}{2} \div \frac{1}{2} = 1$	$\frac{1}{2} \div 2 = \frac{1}{4}$
$12 \div 1 = 12$	$\frac{1}{2} \div \frac{1}{4} = 2$	$\frac{1}{2} \div 4 = \frac{1}{8}$
$12 \div \frac{1}{2} = 24$	$\frac{1}{2} \div \frac{1}{8} = 4$	$\frac{1}{2} \div 8 = \frac{1}{16}$
$12 \div \frac{1}{4} = 48$	$\frac{1}{2} \div \frac{1}{16} = 8$	$\frac{1}{2} \div 16 = \frac{1}{32}$
etc.	etc.	etc.

The analysis which should accompany the presentation of each series of relationships includes a search for the *reason* for the effect of the operation on multiplicand and dividend. In A, we have a constant (6), and we want to find how many are obtained by combining the group an indicated number of times or by taking part of it. In B, the constant is the fractional multiplier; hence, the total obtained must always be less than the part represented by the multiplicand. In C, the constant is the multiplier (4), and as a consequence the product is always larger than the fractional multiplicand. In D, the question is, "How many 4's, 2's, $\frac{1}{2}$'s, $\frac{1}{4}$'s, are there in 12?"; in E, "How many 1's, $\frac{1}{2}$'s, $\frac{1}{4}$'s, etc., are there in $\frac{1}{2}$?", and the quotient must steadily become larger; in F, "How large a part do we get when we divide $\frac{1}{2}$ by 1, by 2, etc.?", and since the divisor is 1 or more than 1, the quotients must steadily decrease in size.

When the computations are viewed in these terms, the operations make sense, the answers can be checked for reasonableness, and the student is not likely to accept such absurdities as: $\frac{1}{2} \times 8 = 16$, or $8\frac{1}{2}$; $\frac{1}{4} \div 8 = 32$, or $8\frac{1}{4}$; $10 \div \frac{1}{2} = 5$, or $9\frac{1}{2}$; and so on. Without the needed understandings, the answers just listed are as reasonable as any others.[8]

Problem solving.

Practical situations in which one computes with common fractions may be grouped under two headings: (1) those resembling the corresponding situations in which whole numbers are used and (2) those involving the special understandings contained in the "part-whole" relationship. Type (2) is fully discussed later (Section F). With respect to type (1) little needs to be added to the foregoing paragraphs. The student who has rich concepts of fractions and who understands the fundamental operations and their effects upon the values dealt with,—such a student

[8] The high school teacher will recognize this as an excellent opportunity to teach the variation in a function as one of the variables increases or decreases in value.

should face problems as contrasted with abstract computations with considerable equanimity. He needs only to be provided with ample practice and with many varied experiences in using common fractions, in order to complete his learning. What has been said previously in this connection with regard to whole numbers, applies equally well here.

Suggestions for teaching.

Instruction in fractional computation, like that with whole numbers, should start with concrete representation,—real objects (e.g., apples) which can be divided and combined in various ways, symbolic materials (e.g., paper cut-outs), or drawings and diagrams. There is no fractional computation which cannot be objectified in some manner. The advantages which accrue from objective presentation are those of clarity and of verifiability. With the beginning of instruction on each skill (and for some time thereafter, at increasingly longer intervals) abstract fractional relationships ($\frac{2}{3}+\frac{1}{4}$; $4\frac{7}{8}-2\frac{1}{6}$; $\frac{5}{8}\times\frac{3}{6}$; $\frac{4}{5}\div\frac{2}{3}$) should be translated into some type of concrete form, and solutions should be arrived at on this basis. After this, they may be translated back again into abstract terms and the answers found through the use of numbers.

E. Operations with Decimal Fractions[9] (Items 14–17)

Computation and problem-solving with decimal fractions differ little from computation and problem-solving with whole numbers. With both, the separate digits of the numbers must be written with rigidly careful attention to place-value, no more so for decimal fractions than for whole numbers. It is only because we customarily omit the decimal point in writing whole numbers (and because, unfortunately, we substitute rules for understanding in placing the digits of whole numbers for computation) that something new seems to be added when we compute with decimal fractions (including mixed decimals).

Usually, addition and subtraction occasion no difficulty when the student deals with decimals, if for no other reason than that he has for years added and subtracted money numbers even though he does not recognize these as mixed decimals. Nor do multiplication and division occasion difficulty so long as multipliers and divisors are whole numbers. Trouble appears, however, when multipliers and divisors are decimals; then come errors in pointing off. The trouble is sometimes met by prescribing rules: in multiplication, add the number of places in both factors and point off accordingly in the product; in division, subtract the number of places in the divisor from those in the dividend, and point off this number of places in the quotient; or, clear the divisor of decimals by multiplying by a power of 10, multiply the dividend by the same factor, and divide as with a whole number divisor.

These rules have their place, but they do not cover all essential learnings. At the least, they should be derived by setting up series of relationships, as in the case of common fractions, like the two series below:[10]

A.		B.
	$3\times 8=24$	$16\div 4=4$
	$.3\times 8=2.4$	$16\div .4=40$
	$.03\times 8=.24$	$16\div .04=400$

These analyses can be given a firmer foundation if the more familiar common fractions are first substituted for the decimals in the series above, and if the operations are performed to show what the values of the answers must be. Once the understandings have been developed on a fractional basis, decimals may be re-sub-

[9] There is no corresponding section for "Operations with per cents," for we rarely add, subtract, multiply, and divide per cents as such, and when we do, we compute with them as with whole numbers. Uses of per cents are pretty well limited to the "part-whole relationship" which is treated later (Section F).

[10] Here again is an opportunity to show variation.

stituted, the answers found, the rules deduced, and the figures in the answers pointed off accordingly.

That computation and problem-solving with decimal fractions is an unsolved mystery for many students is easily established. When students must labor over such simple examples as $.64 \div .32$ (which should be solved by inspection) and when they complacently accept the absurd answers which their unenlightened computations yield them,—when these conditions prevail, it is clear that instruction in decimal fractions must be radically re-oriented. Decimal fractions must themselves be much better understood (this need is discussed above in Section A), and this understanding must be based upon (1) the ideas involved in common fractions and (2) upon an understanding of place value. There is then little required except to make students competent in abstract computation and in solving problems with decimals.

F. Part-Whole Relationships (Items 18–20)

Part-whole relationships in mathematics are of three kinds: (1) to find a part of a quantity; (2) to find what part one quantity is of another; and (3) to find the whole quantity when a part and its relative size are known. Teachers will recognize here the familiar three cases of percentage, the first two of which are commonly taught, the third less commonly. These three part-whole relationships, however, are by no means restricted to percentage. Finding 40% of 60 is no different essentially from finding .40 of 60 or from finding $\frac{2}{5}$ of 60. That is to say, the three expressions denote the same idea. Indeed students who well understand common fractions, decimals, and per cents will choose that form which leads most easily to the answer. In any case the solution is the same whether $\frac{2}{5}$ of 60, $.40 \times 60$, or 40% of 60 is found no matter in which form the problem is stated. Similarly, finding what per cent 6 is of 12 is, in its basic meaning, no different from finding what decimal part 6 is of 12, or what fractional part it is; and finding the whole quantity when 5 is known to be 30% of a number has the same meaning as finding the whole quantity when 5 is known to be .30 of the whole or 3/10 of the whole.

The custom of postponing all consideration of part-whole relationships until percentage is taught and then of limiting these relationships exclusively to percentage is a mistake. One of the most common uses of fractions occurs in such examples as: $\frac{1}{4}$ of 4 is ?, $\frac{3}{8}$ of 24 is ?; and the relationship here is of the first type (Case 1) whether we so designate it or not. In similar fashion we actually use common fractions in school before per cents are taught, in part-whole relationships of the second and third types. What is needed is to recognize these facts and to re-organize the teaching so that this three-fold interpretation of part-whole relationships can be unified. Thus the illustrations given of the three types of part-whole relationship have all been of the common form, when the part to be found, or part of the quantity to be found, or the part known is less than 1. But part-whole relationships also occur of course in which these factors are more than 1. For example, we may want to know 5/2 (2.5 times, 250% of) a number, or to find what part (fractional, decimal, or per cent) 75 is of 30, or to find the whole, knowing that 8 is 5/4 (1.25 times, 125% of) the number. Instruction needs to be given with these latter kinds of example, as well as with the first mentioned. In the discussion which follows, comments are, however, limited to the first kind of example, purely in the interest of brevity.

Finding a part of a quantity.

As has already been said, the student first encounters this particular part-whole relationship in the fractional form of expression. Having found $\frac{1}{5}$ of 10, $\frac{1}{4}$ of 16, etc., many times, he has little difficulty in recognizing the relationship as one of division. The extension to $\frac{3}{8}$ of 20, $\frac{7}{8}$ of 24, etc., should be natural: division is still implied; but the substitution of the sign $\times$ for "of," as in $\frac{3}{5}\times 20$ and $\frac{7}{8}\times 24$, seems to change the idea to that of multiplication only. Actually, the fundamental idea remains that of division. Consider the solutions:

(a) $$\frac{7}{\overset{}{\underset{1}{\cancel{8}}}}\times \overset{3}{\cancel{24}}=21,$$

wherein we divide 24 by 8, then multiply the quotient by 7; and (b) $7\times 24=168$; $168\div 8=21$, wherein we multiply first, and then divide.

When the student begins work with decimal fractions, he again meets what we may call Case I, in such examples as: Find .75 of 20. It should be at once pointed out that he has already solved many examples of this kind and that only two aspects of the situation are new: (a) the operation is performed with decimals instead of with common fractions, and (b) it is performed by multiplication. True, $\frac{3}{4}$ of 20 and $\frac{3}{4}\times 20$ do not look like $.75\times 20$ and $\times .\frac{20}{75}$, but the result is the same because we are multiplying by a number less than 1. This relationship is easily established if a few examples are solved first with common fractions, then with decimals.

Last of all, Case I is met when percentage is introduced. By now the nature of the part-whole relationship should be entirely clear, so that it should be necessary only to show the correspondence between (a) the percentage expression and the solution by per cents (hundredths) and (b) the familiar procedure with decimals.

Finding what part one quantity is of another.

When the part-whole relationship takes the form, "5 is what part of 6?", we are really dealing with the second fraction idea which was developed in Section A, namely, with the idea of the fraction as an indicated division. If this idea is well taught as part of the meaning of the fraction, the part-whole relationship in Case II is already about understood,—but only with the fractional form of stating the relationship. When operations with decimal fractions are taught, it remains to show the equivalence of the new form, "5 is what decimal part of 6?" to the more familiar fractional form of expression. The first few examples of Case II with decimals might well be solved by common fractions until the equivalence is clearly grasped and the student realizes that he is learning, not a new idea, but a different way of dealing with a familiar idea. So taught, the student does not then come upon Case II in percentage as a strange notion. Instead, he can relate this use of per cents to the corresponding use of decimal fractions.

Finding a quantity, given a part and its relative size.

Of the three part-whole relationships, Case III is met with less frequently than are Cases I and II. For this reason, and also because it is "hard" for students, Case III is much less commonly taught. For the minimum needs of the Army (and of civilian life) Case III is not absolutely essential; but the relationship, in all probability, would be much more widely used if it were learned in a meaningful and useful manner.

If the relationship is first presented with common fractions, e.g., "If 6 is $\frac{2}{3}$ of a number, what is the number?", the procedure is that of unitary analysis. Since $\frac{2}{3}=6$, $\frac{1}{3}$ is 3 (because $\frac{1}{2}$ of $6=3$); hence, $3/3=9$ (because $3\times 3=9$). This procedure requires the understanding of simple equations, but this obstacle is not insuperable.

The method of unitary analysis does not seem to be so helpful if one tries to solve Case III examples directly by decimal fractions, as, for example, in: ".5 of a number is 8; what is the number?" However, if the student substitutes common fractions,

$$\tfrac{50}{100}=8;\ \tfrac{1}{100}=\tfrac{8}{50};\ \tfrac{100}{100}=100\times\tfrac{8}{50}, \text{ or } 16,$$

in these circumstances the method of unitary analysis is seen to be applicable; but this does not necessarily lead to the understanding that the solution of $8 \div .5$ is the same thing. For this reason, with both decimal fractions and per cents, it is suggested that the algebraic method be used, letting x stand for "the unknown number." Then the solution takes the form

(a) $.5x=8$ or (b) $50\%x=8$

$.1x=1.6$ $1\%x=.16$

$x=16$ $x=16$

G. Ratio and Proportion
(Items 21–22)

The inventory of Army needs revealed that, as in civilian life, the concept of ratio has wide application. The soldier is confronted with many situations like the following: "25% of the gasoline must be held in reserve," "10% of all parachutes on hand must be kept packed," and "The recipe will provide for 25 men and must be increased in proportion to the number in the mess." These are but a few examples of the many problems in which the idea of ratio is explicit or implicit.

Without a clear concept of ratio, proportion cannot be understood. The ratio idea, however, seems to be difficult for many students, perhaps because ratio involves the measurement of one amount in terms of another amount and the comparison is made by division. Comparisons made by finding differences between amounts are obviously less complicated. For example, to say that "4 is 8 less than 12" seems to be more readily understood than to say that "the ratio of 4 to 12 is 4/12 or $\frac{1}{3}$." Yet, this difficulty could be materially reduced if the idea of ratio were not postponed as long as is common, namely, to the time when ratio is used in computation. If, instead, the idea of ratio were developed as one of the meanings of common fractions (see Section A), the student would be better prepared.

To have competence in the use of ratio and proportion the student must know that: (1) a ratio is obtained when two like quantities are compared by division; (2) ratio can be expressed in different ways; (3) equal ratios form a proportion, and proportions help in the solution of many practical problems.

(1) Ratio arises from a comparison of two like measures, inches with inches, feet with feet, quarts with quarts, dollars with dollars. The unit in which the comparison is made must be the same for all terms. Implied in these statements are two facts: (a) only like measures or units can be compared, and (b) the resulting ratio is an abstract quantity. For example, the ratio of 2 ft. to 10 in. is determined by expressing both terms first in inches; it is then expressed as 24/10 or 12/5: and these numbers have nothing to do with feet and inches. Likewise, the ratio of 12¢ to $1.44 (changed to 144¢) is 1/12, and this ratio has nothing to do with cents and dollars.

(2) A ratio can be expressed in a number of different ways. For example, the ratio of 2 ft. (24 in.) to 10 in. can be expressed as 12/5, as 2.4, or as 240%, and these forms are equivalent. Likewise, the ratio of 1 gal. (8 pt.) to 1 pt. is 8/1, or 8, for ratios may be abstract whole numbers. The student must have experience with ratio expressed in all these forms. Although the fractional form is adequate for all problem situations, the student should know that the ratio 1/2 can be written in the form 1:2. An extension of the idea is that many simple formulas represent ratios between two variables and that the statement of direct variation as in $y=kx$ indicates that the ratio of y to x is the constant k.

The first step in teaching ratio is the comparison of two amounts which can easily be separated or measured off into discrete units. A start may be made with the problem of dividing 15¢ (3 nickels) so as to give one person twice as much as the other (a ratio of 2 to 1); then of dividing 20¢ in a ratio of 3 to 1, of dividing 25¢ in a ratio of 4 to 1, and so on, the amounts being conveniently measured off in terms of nickels.

The same procedure may be repeated with the division of groups of other objects, with lengths, and the like. The logical step to follow is that of indicating the fractional parts of the whole which result, as for example, 2/3 and 1/3 for the 2-to-1 ratio, 3/4 and 1/4 for the 3-to-1 ratio, and 4/5 and 1/5 for the 4-to-1 ratio. These relationships may be stated somewhat differently as: if the ratio of parts is 2 to 3, then the ratios of the parts to the whole are 2 to 5 and 3 to 5. One part is 2/5 of the whole, and the other 3/5. To divide anything according to the ratio 2:3, we must divide it into five equal parts. Ability to think in terms of parts and the whole is fundamental.

(3) A proportion is an expression of equality between two ratios, and because a fraction has many equivalent forms, any two equivalent forms constitute a proportion. Thus, 1/3 equals 3/9 equals 4/12 equals 5/15, etc. Common ways of expressing proportion are: "The more men put on the job, the greater the amount of work done;" "To increase the amount of the mixture, proportional amounts of the ingredients are needed;" "To enlarge or reduce the size of the picture, the width and length must be increased or decreased correspondingly," etc. In teaching the uses of proportion for minimum Army needs there is no need for such terms as *extremes*, *means*, *third proportional*, and the like. However, the expression of proportion in the form $x/4=7/6$ should be known, and the ability to read this form as "x is to 4 as 7 is to 6" frequently supplies a convenient framework for the solution of a problem.

Solving proportions with the aid of literal notation should be stressed. The idea and the skill can be developed by many simple and varied applications, of the types:

(1) If 2 in. represents 13 mi., what does 5 in. represent?

$$\frac{x}{5}=\frac{13}{2}.$$

(2) 400 m.p.h. is how many ft. per sec., it being known that 30 m.p.h. is 44 ft. per sec.?

$$400/x=30/44, \text{ or}$$

$$\frac{x}{400}=\frac{44}{30}.$$

Many simple formulas, as in the expression of direct variation $y=kx$, state that the ratio of the variables, y to x, is a constant. Thus in the formula $d=rt$ when r is fixed, the ratio of d to t is equal to the constant r. The student should recognize the fact that, if $d'=5d$, then $t'=5t$. In a similar manner, without the need for extensive use of the language of inverse variation, the student should see that, if the distance is constant, the rate increases as the time decreases. This idea has been presented earlier in the tables showing the variation when numbers are multiplied and divided (see A, B, C, and D on page 263). A table for a similar purpose may be made for a fixed value of d (say 600) as r takes the values 10, 20, 30, 40, 50, etc. It should be noted, that r becomes twice, three times, . . . as great, t becomes one half, one third, . . . as great.

If time permits, other formulas which will repay the same careful type of analysis are: (1) the law of the lever $L_1/L_2=W_2/W_1$, where W_1 and W_2 are the weight and the force, respectively, and L_1 and L_2 are their distances from the fulcrum; (2) the formula $D/d=n/N$ for expressing the relation between the diameters of two pulleys or gears and their speed of rotation; (3) the formula for an inclined plane $F/W=h/d$ which shows that the ratio of the force F to the weight W is the same as the ratio of the height h which the weight must be raised to the length d of the inclined plane.

The teacher will readily think of other practical uses of proportion which can be used to make the work concrete and interesting, as well as the use of the trigonometric functions to show the slope or grade of a road, the ratio of lead to height in setting guy wires, and others.

H. Powers and Roots[11]
(Items 23–24)

Minimum Army needs with respect to powers and roots consist in: (1) understanding the symbolisms which indicate powers and roots; (2) knowing what powers and roots of numbers are; (3) use of some algorism for finding square roots; and (4) skill in the use of tables of squares and of square roots.

(1) Understanding of the symbolisms for powers and roots, the exponent and the radical sign must be developed so that the student will be able to use them in formulas. He must know that the exponent is an abbreviation for multiplication when the factors are all alike. Adequate practice should assure that the student distinguishes between such expressions as $2a^3$ and $(2a)^3$. Particular attention should be paid to powers of 10 and to the square root of such even powers of 10 as 10^2, 10^4, and 10^6. In this connection, finding the product of decimal fractions by powers of 10 is a good kind of practice. Such practice enriches the student's concept of decimal notation and provides a useful short-cut in computation.

(2, 3) In finding roots, particularly square roots, it is essential that the student have a concept of root so connected with his other mathematical experiences as to be subject to easy recall. This seems to imply a definition related logically to the algorism for finding a square root. The concept of the square root of a number as one of the two equal factors of a number meets the need for definition; and this concept is logically related to finding square root by division.

Thus, to find the square root of 20, one must find two equal factors whose product is approximately 20. The number 4 is too small since $4\times4=16$; the number 5 is too large since $5\times5=25$. So we try 4.5. Dividing 20 by 4.5 gives a quotient of 4.44. The factors (approximate) of 20 are still not equal, being 4.44 and 4.5. But we know that the required equal factors lie between 4.44 and 4.5. So we try as a divisor the average of the two numbers 4.44 and 4.50. Thus the definition of square root is closely connected with the process for finding a square root and a logical bond is established to aid in retaining the concept.

(4) As they are usually printed, tables are serviceable for finding roots and exact powers for the numbers 1 to 100. For larger or smaller numbers, and for mixed decimals even within this range, tables are not so useful. For example, tables do not immediately show the exact square of 8.3; they show only that the square must lie between 8^2 (64) and 9^2 (81), nearer the former than the latter. To find the exact square of 8.3 in a table one must know that $8.3^2=83^2\div10^2$. Similarly, to find the square root of 0.83 in a table, one must know that $\sqrt{0.83}=\sqrt{83}\div\sqrt{10^2}$.

These illustrations show (a) that the usefulness of tables for finding roots and powers, except the simplest, can be easily exaggerated and (b) that the skill required to use tables for these purposes is on a higher level than is commonly thought.

I. Graphs and Maps
(Items 25–28)

The instructors at all camps visited emphasized the need for the ability to read grids and maps as one of the most important of all the mathematical needs of Army men. Strong emphasis was also placed upon the ability to interpret informational charts and simple line graphs. These needs are, of course, quite in line with the rapidly increasing importance of understanding simple graphic techniques in normal civilian life.

What are some of the reasons for the importance given to graphs and maps by training officers in the Army? One instructor made the statement, "There is not a

[11] From this point on, the discussions of teaching suggestions for most topics are relatively short. It is assumed that secondary school teachers of mathematics are more familiar with these briefly treated topics than with those related to the teaching of arithmetic. On this account it is unnecessary to provide detailed recommendations.

soldier in any branch of the service who will not have to know how to read grids and maps. In fact, the life of any soldier might at some time depend upon that ability." Soldiers must be able to follow routes that have been plotted on maps; they must be able to act in accordance with such directions as: "You are now at (104.6, 217.3). You will proceed to (110.0, 223.8)." They must be able to determine the location of certain critical positions, such as the place where mess is to be served, from a plotted line of march.

In order properly to interpret distances pictured on maps, they must understand the basic principles of scale drawing and know how to interpret and use the representative fraction (R.F.) for a given map. They must know that the numerator of any given representative fraction stands for a certain unit on the map while the denominator stands for a corresponding number of units on the terrain mapped. The two representative fractions in most frequent use are 1/20,000, and 1/62,500. Here one unit on the map equals 20,000, or 62,500 units on the ground; these are the actual working approximations of the scales 1 foot=4 miles and 1 foot=12 miles, respectively. Another type of scale that needs to be understood is of the form 1 inch=1 mile, or 1 inch=5,280 feet. Then, too, there are times when it is necessary to estimate heights of certain points of terrain from a map showing contour lines. Maps are also important aids in artillery fire. Targets may be located on a map; approximate hits may then be plotted and proper corrections made.

Protractors and dividers are frequently used as aids in map reading. Dividers may be used in determining distances or in locating specified points. Protractors are important in determining directions.

As stated above, the simple line graph may be employed to portray the line of march of a body of troops. The broken line graph can be used in figuring times of arrival and distances covered or to be covered in troop movements. For example, if a column of infantry is proceeding at the rate of two and one-half miles per hour, a plotted line of march will locate it at any given time or will tell at what time it will clear certain points along the route.

Required less frequently are the abilities to read direct or alternating current graphs and to interpret the impulse graph of a teletype signal. Although not every soldier will need these abilities, it should be noted that, for those who do use them, they are highly important.

Any prospective inductee can profit from instruction in the reading of graphs and in constructing some of the simpler forms. By the time a student has reached the junior or senior high school he should have been impressed with the importance of graphs. From experience, both self-obtained and teacher-provided, he should have become acquainted with their wide use in purveying information. He probably will not know of the two major dangers of misinterpretation of graphs, and he will need to be instructed with respect to them. (1) Ignorance of simple basic facts in the construction of graphs may cause one to read inaccurately the information which the graph accurately presents. (2) Cleverly constructed graphs can be used deliberately to deceive the careless reader or the reader uninformed in the art of interpreting graphs.

Among the basic facts, simple yet important, in the proper reading of graphs, only three can be treated here. They are as follows:

(1) *The student must know what scale was used in the construction of the graph.* In reading any kind of graph (pictograph, bar graph, etc.) he must know what unit is used. In Cartesian graphs he should understand the frame of reference made by the horizontal and vertical axes and the conventional cross-hatching which is merely a convenient aid to accurate graphing. He should appreciate too the fact that different scales may be set up on the two axes, and should know the basic considerations (such as height and spread of graph, rela-

tive size of units, location of the zero-line or base-line, etc.) which enter into the choice of these scales. He should have a thorough understanding of the use of ratio in determining the best scale to use. This use of ratio, of course, should have been acquired in the study of scale drawing (Section O), and should carry a clear-cut understanding of the representative fraction (R.F.) and its implications.

It should be clear that the understandings mentioned above with respect to graphs are closely associated with those required in map reading. The student should have some opportunity to learn how to approximate distances from the use of maps. The use of highway maps in planning vacation trips and estimating distances traveled could be used to provide very good exercise in such use of scales.

(2) *In graphs that are used to make comparisons the student must know how the comparisons are to be made.* Bar graphs and pictographs are most frequently used for this purpose. Comparisons are not usually difficult to make in the case of bar graphs, for only lengths are to be compared. There are occasions, however, when one is interested in more than a mere relative contrast between the given items. For example, certain lengths, such as the distance necessary under normal conditions to stop a car traveling at different speeds, may be compared in a horizontal bar graph.

(3) *The student should know that line graphs are used to show trends and that conventionally a rise from left to right indicates an increase and that a fall indicates a decrease.* In this connection the following are important: the selection of horizontal and vertical scales; ability to detect increases or decreases and to compare changes over corresponding intervals of time; and the location of the zero-line or base line. Without these understandings proper interpretation of line graphs is improbable.

Up to this point the emphasis has been upon the ability to read and interpret graphs. Very little will be said here about the construction of graphs. A high degree of skill in constructing graphs cannot be listed among the minimum needs of the soldier. However, the student can better appreciate the use of graphs if he knows something about how to construct them. To meet the requirements for the minimum needs presented in this report, the student should have experience in constructing straight line graphs, simple broken line graphs, horizontal and vertical bar graphs, and some very simple circle graphs.

As a final word in this connection, it should be added that fairly often soldiers need to make rough map sketches or drawings. Free-hand drawing, in cases where form is wanted and accuracy is not important, would be an aid toward developing this desired ability. At times it is valuable to follow the making of such rough sketches with the estimation of lengths and possibly of areas represented. Sometimes too these estimates should be checked against the already known values of these lengths and areas or against approximations obtained from more carefully constructed sketches.

J. Tables
(Item 29)

Basically one secures information from a table in about the same way as from a graph. The likeness arises from the fact that in both cases an item of information is found at the intersection of a vertical line, whose position is determined by a horizontal scale, and a horizontal line, whose position is determined by a vertical scale. There is also similarity in the procedure in the inverse problem of finding a number which in both tables and graphs is usually represented in the vertical scale. The skills used are described in Section I.

Tables which seem to be of greatest importance to soldiers are: firing tables, such as those for calculating gun elevation and for making correction for windage; tables of conversion, such as those which give the equivalent values of common and decimal fractions and which indicate the equivalence of degrees and mils; and calculating tables, such as those giving squares, cubes, and square roots. The frequency with which such tables are used in the Army makes it desirable for the inductee to be able to deal with them effectively.

The ability to read tables is as important for civilian life as for Army life. It is therefore an objective for general education fully as much as it is an objective in pre-induction training. It is increasingly common for newspapers and other periodicals to present data in tabular form for the enlightenment of the public.

The tables which the enlisted man and the average citizen are expected to use are usually simple in structure, and the facts or relations to be obtained therefrom customarily are rather easily found. It follows that the abilities to be developed are not difficult to teach. The principal problem is one of assuring a reasonably proficient use of mechanical skills. Speed and accuracy in this skill will result from significant experiences with tables, when provided under conditions which reward accuracy and penalize haste.

Up to this point nothing has been said about interpolation in tables, save in tables of roots and powers (Section H preceding). In this earlier place it was stated that when exact figures (roots and powers) are needed, tables are helpful only to the student who possesses, in addition to the simple skill of reading tables, a considerable degree of insight into the theory of roots and powers as this operates in decimal notation. Here it is sufficient to add that ordinarily the requirements of the practical need are met when the student rounds off to tabular values. As a matter of fact, values in tables frequently involve more significant digits than the data of problems justify using. Those who are to use calculation tables should know that in such cases a great deal of futile effort and many unnecessary sources of error can be avoided by rounding off the table values to the number of significant figures that is warranted.

K. Formulas and Equations

(Items 30–33)

In the Army the minimum requirement for algebra is not much more than the ability to substitute numbers in formulas and to solve very simple equations. In general this means that an understanding of the symbolism of algebra and its use is about all that is necessary. But this symbolism should be learned meaningfully.

Formulas.

The use of symbolism varies from substituting numerical values in such simple formulas as $N=\frac{3}{4}A$ to fairly complicated formulas such as $N=R^3KC$ plus a per cent of N which depends on the size of N, a basic formula used by demolition units. Similarly, in civilian life number symbolism is spread over a wide range of usage and includes the rather complicated formulas of the business and industrial world. Familiarity with the basic principles and the characteristic simplicity of such symbolism certainly is to be classed among mathematical needs, not only of men in the Armed Services, but of every intelligent individual.

Abstractness is an inherent characteristic of all symbolism, no matter how simple. When a student sees the symbol r, for example, he needs aid from some source to enable him to determine whether the number r represents "number of units in radius," "number of units in rate," or "number of ohms in resistance," or any one of many possible measurements. All symbolism should take meaning immediately it is associated with the context from which it is derived. It then follows that symbols, which are to be used to describe a certain situation, should be, where possible, chosen to correspond closely with the legend of the situation. For example, if it is *cost*, *margin*, and *selling price* with which the situation is concerned, then c, m, and s, are the symbols to be used; if it is *area*, *length*, and *width*, then A, l, w. However, teachers should make it very plain that these letters are not abbreviations of words, but are symbols for numbers.

There are certain standard formulas which are of sufficient general importance to justify their being committed to memory. From the point of view of minimum needs of the armed services, the most important formulas to be memorized are the area formulas for circle, triangle, and rectangle; the volume formulas for cylinder and rectangular solid; and the formula for distance, rate, and time.

No treatment of formulas can be considered satisfactory unless some attention is given to the concepts of *constant*, *variable*, and *dependence*. From the point of view of minimum needs it is sufficient to develop the distinction between *constant* and *variable* with emphasis on the absence and presence of change, and the concept of *dependence* with emphasis on the fact that the value of one variable may depend on the value of another. For example, the distance traveled by an automobile going at an average rate of 35 miles per hour depends upon how long it travels. There are many such simple examples which can be used from life experiences to bring out the concept of dependence as expressed in a formula.

The principal use of any formula is, by substituting certain given, or determined, values of independent variables, to determine corresponding values for the dependent variable. This understanding seems to be a fairly difficult one to acquire. Part of this difficulty may be attributed to inability to perform the fundamental operations of arithmetic, and part to the failure to comprehend the full significance of symbolic representation. Many students who fully comprehend that the symbol 18/6 means that the number 18 is to be divided by the number 6, are unable to interpret a/b as a symbol meaning some number represented by a divided by some number represented by b. This understanding and appreciation may be developed if the student is given experience in setting up formulas to describe familiar situations.

Equations.

As a natural outgrowth of substitution in formulas there arises the solution of equations. For example, if it is known that at a certain height from the ground the circumference of a tree measures 13 feet, the diameter of the tree can be found by solving the equation $13 = 3.14\,d$. The only equations which need receive systematic attention in this outline of minimum needs are the six types of simple linear equations given on page 249.

In developing the concept of what an equation is and how it is to be used, the idea of *balance* should be stressed. An equation can be defined as a statement that two quantities or expressions are equal in value. With this definition it should be fairly simple to point out the necessity for maintaining balance in order to maintain equality. Need for balance may be used as the basis for the meaningful development of the axioms for addition, subtraction, multiplication, and division. Transposition should not be introduced as a technique at all. It might be allowed as a short cut for those who detect it and who fully comprehend the significance of what they are doing. The four axioms should be recognized as no more than corollaries of the more general axiom: "Whenever the value of one side of an equation is changed in any manner whatever, the value of the other side must be changed in the same, or an equivalent, manner. Otherwise, the balance will not be maintained and the equation will be destroyed." This principle should be kept in the foreground, when dealing with equations.

For students who have difficulty in determining what operation is needed to transform equations, the following considerations may be used as guides:

(1) The four fundamental operations of arithmetic may be related in two pairs of *inverse* operations, namely: addition and subtraction, multiplication and division. Any result obtained by a given operation can be *undone* by using the corresponding inverse operation.

(2) To solve a linear equation in one unknown the equation must be transformed so that the unknown will be *alone* on one side of the equation. To do this we must eliminate all other quantities on that side of the equation.

(3) To eliminate these quantities we must *undo* the operations which associate them with the unknown. For example, in $x+4=12$, 4 has been added. Hence to undo this operation, 4 must be subtracted from both sides. Likewise in $x/4=12$, x has been divided by 4. Hence, to undo this operation, both sides must be multiplied by 4.

L. Positive and Negative Numbers (Item 34)

We are concerned here only with those uses of positive and negative numbers which will meet minimum Army needs. Hence we shall pay no attention to *operations* with such numbers. Men in the armed services do have occasion to understand the nature of directed numbers and their use in indicating opposite qualities and positions. They should therefore know how to establish a zero point and to read positive and negative positions to the right or left, above or below this zero point. They should also know how to read positive and negative angles particularly when related to the north point as zero. Such ability is important in interpreting rifle fire, giving directions, interpreting machine gun firing tables, sighting certain types of guns, and in reading maps and thermometers.

To develop a clear-cut concept of the nature of positive and negative numbers and their characteristics of oppositeness, direction, and position, the most satisfactory approach is to be found in the number scale. Its use should, of course, be supplemented with such familiar concepts as temperatures above and below zero, credits and debits, and income and outgo. The student should not be allowed to think of a negative number as one "whose value is less than zero" and of a positive number as one "whose value is greater than zero." He should understand that a directed number gets its significance through reference to some pre-established zero point, which may be arbitrarily chosen.

M. Measurement (Items 35–40)

Measurement is so important in both Army and civilian life that it merits extended attention here, the more so since the traditional kind of instruction is quite unlikely to produce the needed concepts and skills.

Concept of a unit.

The history of measurement records the early appearance and use of natural units of measure, such as the cubit (length of the forearm), the foot, the furlong (length of a furrow), the hand, and the yard (the distance from the nose to the end of the outstretched arm). Some of these natural units, only part of them conventionalized, are in use today. The child uses his span in playing marbles, his finger width in choosing sides in a ball game, his foot or stick length in playing *cat,* and his Scout staff in woodcraft. Too, the soldier uses his pace in measuring distance and his finger width in placing a distant object, and he sights along the outstretched finger, first with one eye and then with the other, to judge the number in a column of marching men.

These examples of natural units serve two purposes: they show, first, the nature of measurement, as a process of applying some unit as many times as may be needed to obtain the desired measurement, and, second, the nature of the unit of measure. The latter need not be standardized, provided that exactness is not required and that the final estimate achieved does not need to be used by another whose natural units may be different.

Both of these facts are significant for instruction in the field of measurement. To start with the second fact, initial experiences in measuring should be practical and concrete. At this time the units may quite properly be unconventional. Application of these natural units in significant problem situations will reveal the nature of measurement as a process, without complications from considerations of standard scales. Later on, the shortcomings of natural units should be demonstrated and the advantages of standardized units impressed upon the student. At this stage in learning it is worth while to have the student convert some of his natural units into standard units. For the kind of estimating which he will have to do, either as soldier or as citizen, it is valuable for him to know in standard units the length of his pace, for example, and of his span, the width of his finger, and the finger joint which he should use as being nearest in length to an inch. A further value of this process of conversion is that at the same time it increases the meaning of the standard units themselves.

Common units.

Mere knowledge of the names of common units of measure is of little worth, and memorized tables of such names have no more to recommend them. Yet it is not uncommon to assume that the student who can recite the tables of measure has adequate knowledge of the units and their relationships and has adequate ability to use them in practical situations. How far this assumption departs from the truth is easily demonstrated. Let the teacher send six students to the blackboard to draw the true outline of a pail that will hold a quart. The drawings may vary from the size for a pint to the size for a gallon. Or, let students be asked to estimate the weight of a pile of ten books, or the length of a rope, or the distance between two objects a half-mile apart, or the area of the school lawn. Again many errors will be of fantastic proportions, and the range of estimates will be extreme.

As has been already implied, adequate familiarity with units of measure is best attained by *using* them in concrete settings. It is the teacher's responsibility to provide such experiences. One aspect of this provision is to see that the classroom is equipped with the needed materials—foot rules, yard sticks, 25-, 50-, or 100-foot tape, protractors, meter sticks, compasses, dividers, thermometers, scales for weighing, measuring cups marked with ounces, quart and pint measures, and such others as can be secured conveniently.

The other aspect of instruction is that these instruments actually be used, to the end that the student may learn more about units of measure, more about the process of measurement, and more about the skills involved in estimation (see below) as well as the importance of precision in measurement when exact measures are needed.

Applications.

The word *significant* may be used to characterize the problems in connection with which measurement is to be employed in school. Textbook problems are not significant to the average student. Such problems, suitable for practice at a later time largely to insure proficiency in skills otherwise learned, cannot be significant because they do not arise from personal needs on the part of the student. Hence, if textbook problems are used at all in the early stages of teaching, they should serve chiefly as models upon which to frame more realistic situations for measurement.

The classroom offers some opportunities for the significant application of measurement. For certain needs the dimensions of the room itself and of objects in the classroom may be determined. A mimeographed scale drawing of the classroom may be used for practice in reading such representations, the dimensions on the drawing being measured by dividers and rulers and then checked against the actual dimensions of the room. But in so far as is practicable, experience with measurement should take place out-of-doors. By walking, the student can learn the distance of a mile in terms of time. By measuring a plot of ground in the school yard, he may secure a useful concrete way of thinking of an acre.

Such experiences in measuring also give opportunity to use the tape and the transit. The latter need be only a 12″×12″ board to which a protractor has been attached, the whole mounted on a tripod. This homemade transit can be constructed for a few cents in the school shop and is adequate for most needs. One such transit should be provided for each set of four students. Out-of-door practice in measurement with the transit and with other equipment should extend to the solution of problems by scale drawings or the use of trigonometric functions, according to the ability of the class. In some instances this practice must go beyond minimum Army needs if real competence is to be achieved.

Operations with measures.

Civilians and soldiers alike have to be able to add, subtract, multiply, and divide denominate numbers. As in the case of all other abilities, skill is to be had only from practice. Again, as in the case of all other skills, this practice may be furnished too soon, before the student has sound concepts of units of measure and before he understands what happens practically when he adds or otherwise computes with units of measure. Nothing is gained by setting computational tasks with denominate numbers until need for these operations has been demonstrated and until the operations so applied will make sense to the student. Once this stage has been reached, the student is ready to profit in a real way from practice in computing with units of measure. The practice assigned should include some experience in converting units in one scale into units in another scale. At first this abstract practice may well be checked a few times by reference to concrete experience.

Estimation.

As a matter of fact, the soldier and the civilian make estimates far more frequently than they compute with units of measure or actually measure quantities, distance, weight, and volume. For example, every soldier needs to be able to estimate distances between 100 yd. and 600 yd. in order to set the sights on his gun. Soldiers in Ordnance must be able to select at a glance wrenches of various sizes. Soldiers in the Signal Corps must be able to estimate 100-ft. distances between communication poles and the 11- or 16-ft. clearance of wires over different classes of roads.

The ability to make reasonably accurate estimates depends upon sound concepts of units of measure, natural or standard, and upon adequate experiences in applying these units. Experience in estimating will scarcely be adequate if it is limited to the activities which are usually organized within the walls of the classroom. The value of natural units for purposes of estimation has already been suggested, but the value of these units is greatly enhanced if the user is able to convert them at once into standard units. To illustrate, the soldier who in high school has been a track man may estimate distances in terms of the 50- or 100- or 220-yd. dash. One who formerly was a railroad man may estimate distance in terms of box-car lengths. Still another may estimate the sizes of timbers or the diameters of trees by the span or the width of the hand. Such natural units are seldom recognized and taught in classroom instruction, but their usefulness warrants such recognition.

The chances of successful estimation are increased if the student knows mathematical values well and is familiar with common approximate equivalents. He will find it useful, for example, to know that 17/64″ is a little more than $\frac{1}{4}$″; that 0.035 is half a hundredth less than 0.04; that a rivet 0.316″ in diameter cannot enter a $\frac{5}{16}$″ drilled hole. Among the common approximate equivalents which it is well for the student to know are: a kilometer is about $\frac{5}{8}$ mile; a meter is about 3 in. more than a yard; a liter is a little more than a liquid quart; a gram is about 15 grains; an ounce (apothecaries) is about 31 grams; a kilogram is about $2\frac{1}{5}$ lb.; 5 c.c. equal one teaspoonful;

5 centimeters equal about 2 inches.

Limits of accuracy.

When actual measurement is undertaken, the degrees of accuracy required may vary. In other words, accuracy of measurement is a relative matter. An error of $\frac{1}{4}''$ in the measured length of a 40′ piling is not usually serious, but an error of the same amount in measuring the diameter of a howitzer is likely to be disastrous. This concept of acceptable and unacceptable amounts of error in measurement is known as tolerance and as limits of accuracy. In the meaning of these terms in this report there is no implication that the student must be able through elaborate calculation to determine per cent of error or per cent of relative error. What is implied is that the student must have a common sense understanding of the terms as related to the situations in which they are used. An error of 0.001″ in making a machine part may well be a greater error than an error of 1″ in measuring 100′; an error of 1 grain in weighing a drug or of 1 c.c. in measuring a medicine may be fatal in a hospital; an error of 1° in measuring an angle on a scale drawing may make an error of many yards in an indirect measurement.

N. Geometric Concepts (Items 41–43)

The mastery of the geometric concepts mentioned in the outline of essential mathematics cannot be obtained by merely memorizing definitions. For each concept there must be enough understanding to guarantee its usefulness and its actual use. Admittedly, the degree to which understanding needs to be carried varies from term to term. It is enough that the soldier in Ordnance be able to identify elliptical and parabolic cams, but for most of the listed geometric terms much more than mere identification is required. In general, the following should be accepted as objectives for instruction: (1) ability to recognize the form, (2) knowledge of its fundamental properties, (3) skill in representing the form by a drawing, and (4) competence in using the form where and when needed in practical situations.

The degree to which the student is able to identify geometric forms and to retain the concepts which he has learned depends upon the number and variety of relationships he is able, under instruction, to establish. In a word, many contacts are required to develop and fix concepts of geometric forms and their properties. These contacts should not be restricted to classroom work with pencil and paper (a caution which has appeared many times in this report), but should include practice in recognizing geometric forms out-of-doors. For example, a telephone line is not straight, but has a sag. Perpendiculars are visible everywhere, as well as natural forms which are approximately vertical.

Whether a line is truly vertical can be determined by a plumb line or by use of the 3-4-5 relationship to something else which is known to be horizontal. Horizontals, which are so important in carpentry, among other jobs, are also somewhat generally visible inside and outside the school. Tests can be applied to determine whether objects assumed to be horizontal are actually so, by use of the level or again by reference to the 3-4-5 relationship, this time to something known to be vertical or perpendicular. Right and isosceles triangles may be located in many natural forms and particularly in architecture, as can also the forms of the cylinder, the cone, the prism, and the sphere. Regular polygons are found in flowers, in vegetables, and especially in snowflakes. In all instances, when the form is recognized, study should be continued to isolate and identify the fundamental properties. In connection with this study it is possible to discover and learn the size of each angle in an equilateral triangle and in a regular hexagon and octagon. It is also possible in this study, if opportunity arises and the need is apparent, to go beyond the minimum list of geometric concepts mentioned in the outline. The tangent and cotangent ratios are used in the Army even when the names of these functions are not employed. The slope or grade of a road is expressed by the Engineers as the *vertical rise* over the *horizontal distance*; the ratio of *lead* over *height* is used in the Signal Corps in attaching a guy wire to a pole; and men in Ordnance use the sine bar in the shop.

Accurate knowledge of geometric forms obviously includes knowledge of the fundamental properties inherent in their definitions; but it also includes knowledge of other properties which may be deduced therefrom as well as still others derived from such formulas as those for the area of a plane figure, for the perimeter of a plane figure, and for the volume of a solid. These classes of properties may be illustrated. A square is a figure with four right angles and four equal sides. If this definition does not distort logic too much, it follows that the basic properties of the square are the four right angles and the four equal sides. By definition, lines are parallel when, if cut by a transversal, the corresponding angles are equal. This definition contains the fundamental properties of parallel lines; but others may be inferred. If the corresponding angles of parallel lines cut by a transversal are equal, then, since vertical angles are equal, so also are the alternate interior angles. From these inferred properties it follows (a) that the opposite angles of a parallelogram are equal and (b) that the sum of the angles of a triangle is 180°. The foregoing statements do not imply a course in systematic geometry. They show how to prevent facts and concepts from being learned as isolated and unique.

In the teaching of geometric concepts, the activity of drawing, besides developing a skill of importance in its own right (see Section O), is both a good instructional device and a good evaluating device. Drawing aids in learning geometric concepts by forcing attention to the critical aspects of the geometric form or concept in question and by affording practice in the use of protractors, dividers, and compasses. It aids in the evaluation of learning by setting a crucial test. The student who can reproduce geometric forms by drawing from memory has given evidence of learning that is more convincing than is the mere ability to define or to select from suggested alternative definitions.

In conclusion, let it be said once more that geometric concepts are learned when they can be used intelligently. The best preparation for intelligent use is learning and practice in concrete, practical situations, motivated wherever possible by personal need. One last example: (a) the formulas for area and volume can be memorized and immediately employed in computation; or, (b) they can be developed through concrete experience—that for area, by having the student count the number of one-inch squares required to cover a surface; that for volume, by having him fill a box with one-inch cubes. The result of procedure (a) will be superficial, useless knowledge. The result of (b), provided it is employed often enough and with different materials in different situations will be knowledge that will function in new problem situations.

O. Drawing and Construction (Items 44–47)

In all branches of the service the training officers consulted marked as very frequently needed the ability to read blueprints, maps and sketches. In the case of maps, sketches, and grids (coordinate systems), the needed ability, they said, includes that of orientation with respect to compass bearings. In turn, the ability to orient, when the lines of the grid do not show compass north, involves the understanding of positive and negative numbers. (See Section L.) The requisite knowledge and abilities, it needs scarcely be pointed out, cannot be acquired by practice alone in examining completed grids, maps, and blueprints in textbooks. Here again mathematical sense is of critical importance. As is true elsewhere, mathematical sense in connection with maps and grids must be based upon understanding, and the needed understanding is most surely guaranteed by engaging in activities which go beyond the study of products which have already been worked out by others.

In the case of mapping, the scale is frequently given in the form of a representative fraction, R.F. (such as 1/20,000), where the numerator stands for a distance on the map and the denominator for a distance on the ground. The teacher will recognize that the particular R.F. mentioned is somewhat more difficult than the more usual forms, such as 1″=1 mi. (R.F., 1/63,360). (See the discussion of representative fractions in Section I.) Ability to read maps in which R.F.'s are stated includes, obviously, understanding of the particular R.F. employed and understanding of orientation. But instruction cannot stop at this point, because sometimes it is necessary to determine the R.F. from a foreign map when none is given on the map actually being used. In addition, the translation of an R.F. such as 1/63,360 so as to determine the number of yards between two points a certain number of inches apart on the map requires some computation, and practice should be given in this connection.

In the case of blueprint reading, the needed competences are (1) to use the measurement of parts given, (2) to find the dimensions of parts not given, and (3) to visualize the exact shape and form of the part represented. The last named ability is very complex; it includes some degree of abstraction and seems to be dependent on considerable experience both with reading and with making scale drawings.

The equipment needed in teaching drawing and construction within the limits of this report has been partly outlined in the section on Measurement (Section M). Obviously needed are rulers calibrated (1) to thirty-seconds of an inch and (2) to tenths of an inch. A reasonable approximation of the latter type of ruler can be obtained by using graph paper which is ruled 10 squares to the inch. Besides rulers, compasses, dividers, and protractors are essential. Use of drawing boards, T-squares, and drawing triangles will make experiences in scale drawing more valuable and more interesting. These last-named kinds of equipment need not be elaborate and can be made in the school shop or by the student at home. They facilitate the making of scale drawings because of the ease with which parallels and perpendiculars may be constructed. If drawing boards, T-squares, and triangles are lacking, graph paper will again make a good substitute, for the parallel and perpendicular lines on graph paper may be used to good effect. A wooden triangle and ruler can serve the same purpose.

So far as possible, instruction in drawing and construction should be definitely related to problems of real worth. Scale drawings of machine parts and of house plans should be made and studied; orthographic drawings of simple objects should be constructed; and, when feasible, as has already been suggested, blueprints should be studied. Problems requiring the finding of the side or hypotenuse of a right triangle can be set up from real situations arising inside or outside the classroom. In all drawing and construction, whether in such real problems or in practice work assigned to secure proficiency, the student must draw lines carefully so that they actually pass through the required points and draw angles so that they are the correct size. As in all other mathematical learning, the most accurate results are secured if each completed step is tested out by common sense checks.

Although the minimum objectives in drawing and construction are stated so as to include only the ability to read and understand, competence, even of this degree of simplicity, can be assured only by extension of the topics in classroom teaching. Many of the extensions may have but rare application in the Army but they contribute, as learning experiences, to the objectives sought for the skills in question as well as to other objectives discussed in earlier parts of this report. Some of these extensions are: the use of scale drawings to solve problems in indirect measurement by the construction of triangles, or to solve problems by the use of lines which represent force and direction (vectors); geometric constructions (as opposed to drawings made with the protractor and drawing board equipment) for erecting perpendiculars, bisecting angles and line segments, copying angles, and dividing a line segment into equal parts; and the Pythagorean relation. To these extensions may be added for students with adequate capacity, preparation, and interest, the use of tables of sines, cosines, and tangents in solving problems.

P. Miscellaneous (Items 48–49)

Two topics that do not seem to belong in the categories already discussed are treated in this section. They are: measures of central tendency (average, median, mode) and rounding off numbers.

Average, median, mode.

The average (arithmetic mean) is used rather frequently, the median and mode less frequently, in the Army. Mention of these measures of central tendency should not imply systematic instruction in statistical procedure. Problems in finding the average of measures can be made to arise naturally throughout the work of the class. The median is easily taught as the middle measure of an odd number of measures, or as the average of two middlemost measures in an even number of measures. The mode should be found by inspection, from a frequency distribution. As a matter of fact, the median and mode may be found without grouping into class intervals. Instead, measures as given may be listed according to size from largest to smallest (or vice versa), all measures of the same value being grouped together. This simple procedure provides experience in the orderly consideration of numbers, in tallying, and in checking, skills all of which contribute to mathematical sense.

Rounding off numbers.

The student should be taught to round off numbers in such a way that greater accuracy is not pretended than is justified by the data. For example, when the other data in a problem are accurate to one or to two digits only, it is useless and misleading to take 3.1416 as the value of *pi*. In such instances the product may be close enough if the multiplier is 3.14 or even 3.

In rounding off numbers the usual rule is to increase the last retained figure by 1 if the figure dropped is 5, 6, 7, 8, or 9. But instruction on rounding off is incomplete if it includes only the teaching of this rule. The student should understand the reason for rounding off numbers and the occasions when numbers can and when they cannot be rounded off. A measurement of 0.992″ cannot be rounded off to 1″ if a machine part is concerned, but a measurement of 78′ may be called 100′ in setting communication poles if the terrain allows. Here again common sense must be combined with mathematics, and here again instruction must provide experience with practical situations in which the skill to be taught is involved.

Setting Up the Instructional Program

Let it be stated again that the items listed previously must be part of the equipment of all Army inductees. On this point there is no distinction among prospective inductees who are in private schools or in public schools, who have left school for no particular reason, or who have positions in industry. In the age population concerned there is no distinction between prospective inductees who are in high school and those who are in the grades, or, within the high school, between those who are taking the four-year mathematics sequence and those who have had one year or less of mathematics since the eighth grade. This fact carries a number of implications for persons charged with the responsibility for organizing instructional programs.

(1) Students majoring in mathematics[12] need to be examined carefully to make sure that they do have the needed ideas and skills. To the extent that they do not, provision should be made in each advanced course to correct all deficiencies. (And it should be repeated that this provision is inadequate if it consists only in "refresher" practice, to re-instate useless number tricks.) Furthermore, the content of these advanced courses should be studied carefully with a view (a) to eliminating material which will not function and (b) to including vital applications arising from the emergency. In this connection, mathematics teachers should consult the report of the Office of Education Mathematics Committee, in which many valuable suggestions are offered. Also, they might well make use of the Seventeenth Yearbook of the National Council of Teachers of Mathematics (1942), entitled "A Source Book of Mathematical Applications."

(2) As for a special course, to provide students in school with the mathematical essentials, school officials must think in terms of the *ages* of students, not in terms of the *grade* level they may have reached. Physically able 16- and 17-year old prospective inductees will soon be in military service, whether at present they are high school students or are enrolled in the eighth grade. Steps should be taken to bring together in convenient groupings all students who are about to be inducted or are about to leave school for work. It is obvious that the school offers the readiest means of assembling prospective inductees for instructional purposes.

(3) Prospective inductees already out of school, for whatever reason, should be given an opportunity to make up their mathematical deficiencies. It is often possible to have special classes in public school buildings late in the afternoon or in the evening. Many business colleges are already offering such courses, as are many private industrial concerns. These agencies of pre-induction training should be encouraged and multiplied.

So much for the organization of classes from the standpoint of class membership. Next to be considered is a problem which is by no means easily solved but which is not hopeless. This is the problem of determining when students have acquired the necessary mathematical essentials.

This problem seems to be easy (indeed, it has been said by some to be relatively easy). It would be easy if we were concerned only with the traditional outcomes of mathematical competence, namely, abstract computation and problem solving. Many tests in these two areas are commercially available; and these tests may be trusted to reveal what they *can* reveal. For this reason their use can be justified.

But these tests do not, because they cannot, reveal the whole picture. Enough has already been said about the inadequacy of abstract computational skill as a basis for successful adjustment to Army life. The consistent tenor of this report has been that to computational competence must be added what has been called mathematical sense. And there is no commercial test of this outcome.

Yet, the lack of comprehensive and valid mathematical tests does not make the situation desperate. Mathematical sense (and its absence) is revealed under conditions of *use*. The boy who can quickly and accurately size up the quantitative aspects of concrete, practical situations and employ the correct mathematical techniques has passed the test of mathematical sense: he has acquired what the Army (and civilian

[12] There has already been a marked increase in the number of high school students in advanced courses in mathematics, and the trend will undoubtedly continue at least for the duration. Perhaps a word of caution is not amiss. Students should be encouraged to major in mathematics only when they have the capacity for such study. All other students should be guided into special classes based upon the minimum list of ideas and skills above, or upon extensions thereof.

life) wants him to have. The evidence he has given of his learning is wholly respectable, even if it has not been presented through a standard test or a locally prepared objective test. Evidence of his learning is of course secured through observation, and naturally the quality of observation will vary with the competence of the observer. We have assumed a competent observer.

Competence in observation is possible only when the observer knows (1) what he is looking for and (2) what constitutes sound evidence. In the case here in question, namely, in the evaluation of mathematical learning, the requirements for competent observation are readily stated, if not so readily realized. The teacher of mathematics must know (1) the outcomes he is seeking to attain in the behavior of his students and (2) the ways in which the students will behave if they achieve the outcomes set for them. As part of the latter requirement, he must know how to set up situations which will permit the behavior changes to reveal themselves.

Viewed in this way, evaluation is but part of the complete act of teaching. The good teacher evaluates continually,—necessarily so, for he is constantly on the alert for changes in behavior. Changes in the right direction inform him that he is teaching successfully; changes in the wrong direction, or no changes at all, tell him that his teaching is unsuccessful, and that he must try again in a different way.

The committee is not therefore shirking a responsibility when it offers no simple, mechanical method to determine when prospective inductees are mathematically ready for their tasks in the Army. There *is* no such simple mechanical method. The committee has presented a positive, practicable proposal: Let the mathematics course be taught by persons who know mathematics, not alone in the technical sense, but in the sense of concrete, sensible practicable application. Such teachers will be able to tell when their students have acquired the ideas and skills set in the minimum list. The present bulletin must stop where it does stop,—with the specifications for the job. Beyond this point responsibility lies in the hands of school officials and teachers. A careful study of this bulletin, the committee believes, will enable them to meet this responsibility more completely and more wisely.