

Program for College Preparatory Mathematics

Report of the Commission on Mathematics

College Entrance Examination Board

1959

Why?

The formation of the Commission was the outcome of a combination of concerns and interests expressed by a number of Board groups: the Examiners in Mathematics who requested a study of the secondary school mathematics curriculum; the Committee on the Examinations, which identified the need for exploratory actions; and the Trustees, who approved the appointment of the Commission. The Examiners (well-known school and college teachers who were in charge of the Boards' entrance examinations in mathematics) questioned if it was appropriate, in the second half of the twentieth century, for an examination in secondary school advanced mathematics to be committed, in more or less equal parts, to trigonometry, advanced algebra, and solid geometry. The Examiners addressed these questions: "Did the tests accurately reflect what the schools were teaching? What kind of mathematics *should be* studied/taught today? What were the implications of curricular changes to secondary schools?" (p. xi)

What?

The major proposals of the Commission are outlined in the following nine points:

- 1) Strong preparation, *both* in concepts *and* skills, for college mathematics levels of calculus and analytic geometry
- 2) Understanding of the nature and role of deductive reasoning—in algebra, as well as in geometry
- 3) Appreciation of mathematical structure (patterns) for examples, properties of natural, rational, and complex numbers
- 4) Judicious use of unifying ideas—sets, variables, functions, and relations
- 5) Incorporation with plane geometry of some coordinate geometry, and essentials of solid geometry and space perception
- 6) Treatment of inequalities along with equations
- 7) Introduction in grade 11 of fundamental trigonometry—centered on coordinates, vectors, and complex numbers
- 8) Emphasis in grade 12 on elementary functions (polynomial, exponential, circular)
- 9) Recommendation of additional alternative units for grade 12: *either* introductory probability with statistical applications *or* an introduction to modern algebra. (p. 33)

The Commission was appointed (1955) for the acknowledged purpose of improving the program of college preparatory mathematics in the secondary schools. The appointment of the Commission was recommended by the Board, which served as an agency where school and

college representatives were brought together to discuss, develop, and implement policies with respect to the student's transition from school to college. The Commission had no authority to enforce changes; its role was to recommend and to suggest, but **not** to dictate, the immediate future of the secondary mathematics curriculum. The final report was published in 1959.

Who?

The Report of the Commission on Mathematics by the College Entrance Examination Board consisted of fourteen prime movers and distinguished educators who gave life to the Commission by strongly recommending its creation.

Albert W. Tucker, Princeton University, Chairman
Carl B. Allendoerfer, University of Washington
Edwin C. Douglas, The Taft School, Watertown, Connecticut
Howard F. Fehr, Teachers College, Columbia University
Martha Hildebrandt, Proviso Township High School, Maywood, Illinois
Albert E. Meder, Jr., Rutgers, the State University of New Jersey
Morris Meister, Bronx High School of Science, New York, New York
Frederick Mosteller, Harvard University
Eugene P. Northrop, University of Chicago
Ernest R. Ranucci, Weequahic High School, Newark, New Jersey
Robert E.K. Rourke, Kent School, Kent, Connecticut
George B. Thomas, Jr., Massachusetts Institute of Technology
Henry Van Engen, Iowa State Teachers College
Samuel S. Wilks, Princeton University

Among persons of note participating by attending one or more of the Commission's meetings and/or writing sessions were the following:

Max Beberman, Director, University of Illinois Committee on School Mathematics Project
Edward G. Begle, Director, School Mathematics Study Group
William Betz, Supervisor of Mathematics (retired), Rochester, New York
S.S. Cairns, Professor of Mathematics, University of Illinois
Saunders MacLane, Professor of Mathematics, University of Chicago
G. Bailey Price, President, MAA

The commission noted that its findings were the result of the collective wisdom and experience of the assembled members. Consequently, while a report was produced by the Commission, each member brought their own perspective to the discussions. This diversity is reflected in an introductory statement of the final report:

Not all of us in the Commission hold precisely the same views in education in general, or on secondary school mathematical in particular. We represent different backgrounds, diverse professional experiences and a variety of institutional affiliations. Some of us have dealt largely with highly selected groups of students, others with more heterogeneous groups. However we all believe in certain principles

of education, many of which find expression in this report and have been applied in working out our recommendations. (p. xii)

What was produced?

Contents of the CEEB Report

The program for college preparatory mathematics consisted of the following six chapters:

- Chapter 1: Orientation: an urgent need for curricular revision
- Chapter 2: Secondary Education: the Commission's premises
- Chapter 3: Recommendation: the Commission's program
- Chapter 4: Organization: a proposed sequence for the Commission's program
- Chapter 5: Implementation: the vital role of teacher education
- Chapter 6: Articulation: the school and the college

In general at all grade levels, the Commission proposed a collection of mathematical topics with more specific recommendations at grades 9-12:

General:

I) The Arithmetic

- Fundamental operations and numeration—mastery of the four basic operations with whole numbers and fractions, written in decimal notation and in the common notation used for fractions (life situations, place system in writing numbers, binary notation, large and small numbers, approximation of square roots for whole numbers)
- Ratio—understanding of a ratio in comparing sizes of quantities of like kind, in proportions, and in making scale drawings (percent, rate, interest, discounts)

II) Geometry

- Measurement—the ability to operate and transform the several systems of measure, including the metric system of length, area, volume, and weight (length of line segment, perimeter of a polygon, circumference of a circle, areas of regions enclosed by polygons and circles, surface areas of solids, volumes of solids, measure of angles by degrees)
- Relationships among geometric elements—include the concepts of parallel, perpendicular, intersecting, and oblique lines; acute, right, obtuse angles; scalene isosceles, and equilateral triangles; right triangles and the Pythagorean relation; sum of the interior angles of a triangle

III) Algebra and Statistics

- Graphs and Formulas—use of line segments and areas to represent numbers (bar graphs, line graphs, pictograms, circle graphs, continues line graphs, meaning of scale, formulas for perimeters, areas, volumes and percents, symbols in formulas, simple expressions involving variables)

More Specifically:

Mathematics for grades 9, 10, and 11 (Elementary and Intermediate Mathematics)

- **Algebra**—Algebra should be taught with the focus on the development and understanding of the properties of a number field, and not solely the development of manipulative skills. However, it is necessary to note the following statements:
 - 1) The Commission did *not* propose a teaching of elementary algebra from an abstract point of view. It also *did not* propose the teaching of abstraction, unless the student had explored the concrete or intuitive conceptions to build upon. The Commission understood that abstraction would hinder conceptual understanding for the beginner, especially if used as a “point of departure.” The first Appendix, “Introduction of Algebra” (p. 3), illustrated how the “laws of algebra” (axioms of the a number field) are introduced as simple notions of the sets, instead of arbitrary “rules.”
 - 2) The Commission agreed with the importance of teaching appropriate manipulative skills, but these special skills should not be the *only* focus of instruction. The goal of effective instruction is to promote conceptual understanding and “deductive reasoning” (p. 21). The Commission also noted that both skills and concepts were important.

Specific recommendations for Algebra:

- *The New with the Old:* The old—terminology, symbolism. The new—understanding of primary ideas and concepts of the subject (nature of number systems; basic operations and “laws”: commutative, distributive, and associative); the applications of the learned material; the meanings of equations and their solutions; identities; introduction of new concepts dealing with inequalities (both algebraically and graphically); and nature of functions
- *Deductive Reasoning in Algebra:* Deductive reasoning should be implemented in *all* mathematics courses, and not limited to geometry. Deductive reasoning involves the ability to apply the knowledge through understanding of the nature of the subject, in comparison to manipulative skills that promote blind following of rules or techniques to solve a problem.
- **Geometry**—“There are essential defects in the Euclidean development of geometry that made it unsuitable as the basis for modern high school instruction” (p. 23)

Felix Klein pointed out that Euclid’s intentions were not to emphasize mathematical understanding or discovery, or to write an introduction suitable for high school students. Euclid approached geometry to formulate general philosophical theories. “He wrote for scholars not for schoolboys.” Felix Klein compared the purposes of studies in geometry by two scholars: Archimedes and Euclid. Archimedes focused on practical applications and calculations (computed a very acceptable value of π) of mathematics. Archimedes demonstrated how mathematical results are elaborated and conjectured before they were proved. “Archimedes was a creator; Euclid a compiler” (Appendix; p. 109 of Chapter 10 and p. 166 of Chapter 18).

Specific recommendations for Geometry:

- *The number of theorems should be reduced:* In comparison to the traditional geometry textbook, where the students are usually asked to provide formal proofs, the number of basic theorems in a new contemporary curriculum needed to be reduced. There should be 10 or 12 propositions with required proofs as a part of deductive reasoning, all other propositions should be treated as originals.
- *Coordinate Geometry should be introduced:* The Commission felt that, ideally, earlier in the geometry course the student should experience coordinate geometry immediately after the first sequence of theorems. The student would then be able to combine the geometric facts of the sequence with his earlier knowledge in graphical algebra, and perhaps make connections. Once coordinate geometry had been introduced, the Commission urged the use of analytic (algebraic) as well as synthetic methods in proving geometric theorems and exercises.
- *Place of solid geometry:* It is possible and desirable to teach certain material of solid geometry alongside with the parallel content of plane geometry, rather than separating them. Solid geometry should cover: lines, planes, angles, dihedral angles, and spheres.
- *Other Geometries:* Euclidean geometry is not the only geometry that deals with the concept of congruence. For example, spherical geometries (i.e. “non-Euclidean”) allow “the sum of the angles of a triangle that always exceed two right angles to be congruent if corresponding angles are equal” (p. 27). Therefore the Commission felt the need not to emphasize only Euclidean geometry, but also a brief introduction of other geometries.
- **Trigonometry**—“Computational emphasis should shift from triangles to vectors, and analytic emphasis from identities to functional properties” (p. 28). Four trigonometric units were suggested by the Commission (p. 29): 1. Rudimentary trigonometry or right triangles; 2. Trigonometry of x, y, r, θ – coordinates, vectors, complex numbers; 3. Cosine and sine laws, addition theorems, and identities; and 4. Circular measure, circular functions and their wave nature.
- **An Introduction to Statistical Thinking**—Statistical thinking as part of daily activities, and an introduction to statistical thinking in high school will enhance deductive thinking. Numerical data, frequency distribution tables, averages, medians, means, range, quartiles were to be introduced in 9th grade. A more formal examination of probability concepts should be introduced later (grade 12).

Mathematics for Grade 12 (Advanced Mathematics)

Three possible programs were suggested:

1. Elementary Functions, first semester; Introductory Probability with Statistical Applications, second semester.
2. Elementary Functions, first semester; Introduction to Modern Algebra, second semester.

3. Elementary Functions and Selected Topics: Elementary Functions enlarged to a full year by additional topics.

Elementary Functions

- I. Sets and combinations. Ideas stressed: Combinations, permutations, and the binomial theorem.
- II. Functions and relations from a set-theoretic approach.
- III. Polynomial functions.
- IV. Exponential functions.
- V. Logarithmic functions.
- VI. Circular functions, using the wrapping function (NCTM, 1970).

Second Semester Alternatives

The above course on Elementary Functions completes the minimal four-year program proposed by the Commission. If all the material is completed by the end of the first semester of the 12th grade, one of the following three alternatives is suggested for the second semester.

Alternative 1: Introductory Probability with Statistical Applications

This outline was fully developed in the Commission's experimental textbook, *Introductory Probability and Statistical Inference* (revised preliminary ed., New York: College Entrance Examination Board, 1959).

- I. The nature of probability and statistics.
- II. Organization and presentation of data—the frequency distribution.
- III. Summarizing a set of measurements—the mean and standard deviation.
- IV. Intuitive approach to probability.
- V. Formal approach to probability.
- VI. The law of chance for repeated trials—the binomial distribution.
- VII. Applications of binomial distribution.
- VIII. Using samples for estimation—sampling from a finite population.
- IX. (Supplementary) The laws of uncertainty—probability distributions.
- X. (Supplementary) Relations between two variables—fitting a straight line.

Alternative 2: Introduction to Modern Algebra (fields and groups)

- I. Fields.
- II. Ordered Fields.
- III. Abelian (i.e., commutative) groups.
- IV. Transformation, composition.
- V. Groups (not necessarily commutative)

Alternative 3: Selected Topics

The two preceding alternatives each require a full semester. This third alternative is more flexible: topics may be selected for part of a semester or for a full semester.

- VII. Additional work on sets, functions and relations.
- VIII. Further treatment of mathematical induction.
- IX. Further treatment of permutations, combinations, and the Binomial Theorem.
- X. Probability.
- XI. Additional work on inequalities and absolute values, solution sets and graphs.
- XII. Graphing of factorable polynomials and rational functions.
- XIII. Systems of equations, with emphasis on graphical interpretation.
- XIV. Coordinate geometry of three dimensions (CEEB, 1959, pp. 43-47).

Implementation: The vital role of teacher education

Not only did the Commission recommend that any secondary school teacher have four years of mathematics in high school, but also that elementary school teachers should take at least three years. The Commission called on universities to help educate in-service elementary school teachers about the new topics of statistics, modern algebra and mathematical logic, and teach the material to in-service elementary teachers. For in-service teachers, the Commission suggested study groups, seminars, programs of continued study and, again, involvement on the part of the university to aid teachers whose knowledge was incomplete.

Articulation: The school and the college

In this section, the CEEB pointed out that the college entrance examinations would change over time to reflect its suggested changes in the high school curriculum. The Commission admitted that the process would be a slow, but ultimately beneficial to students.

Significance of the Report

The Commission's work was valuable and was carried forward by other groups, which had interest, funds, and the capacity to translate the proposals into courses of study, textbooks, and teachers' guides. The Commission also suggested reference materials, and other requisite tools of instruction that were indispensable in the process of employment of any major curricular changes. The Commission's recommendations were considered with those of other groups to improve the study of mathematics from first grade through graduate school.

Some statements by the Commission, especially those emphasizing that instruction should *not* focus solely on manipulative skills, are as valid today as half a century ago.

The Commission recommended professional development of teachers designed to strengthen their mathematical knowledge.

The knowledge of teaching methods and of educational psychology is no substitute for a sound understanding of such basic mathematical concepts as the number system, and its place in the mathematical scheme of things as well as its uses in every day life.
(p. 48)

The Commission recommended three years of secondary mathematics especially for *elementary teachers* “for successful and assured teaching of mathematics” (p. 49).

For *teachers* this report was also significant in its content. It proposed the design of new courses that would meet the specific needs of the secondary school teachers.

The advanced courses in mathematics are not suitable because in most cases the teachers are returning to the campus after absence of several years, or perhaps even a decade or two. In this period some prerequisite knowledge has been lost and they are no longer qualified to enter to the graduate program they might have been fully prepared for right after the completion of their college degrees. The education courses are equally unsuitable. The secondary teacher’s greatest need in order to prepare to teach the curriculum, is not methodology but subject matter and it’s relevance to the secondary school mathematics s/he teaches. (p. 51)

For *pre-service* teachers the Commission made an imperative statement that the undergraduate programs must be adapted so that future teachers will have opportunities to study contemporary mathematics and be prepared to implement new mathematics.

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