Linking Geometry and Algebra through Dynamic and Interactive Geometry

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Focus of the talk

- Dynamic interactive geometry as a tool for linking algebra, precalculus, calculus and geometry
- Based in particular on two PhD theses done in Grenoble:
  - of Rossana Falcade (2006) about the teaching of function in high school by using dynamic geometry
  - of Julio Moreno (2006) about the use of dynamic geometry by preservice teachers solving tasks on differential equations

Starting claim

- Every mathematical activity is mediated by external representations (Duval, Arzarello, d'Amore, Moreno Armella, Bosch & Chevallard, Kaput & Schorr, Mariotti…)

Flexibility between representations

- Using various representations of a same concept and moving between them are part of the construction process of the concept (Duval)
- Doing mathematics requires processing and acting with and on various representations
- This is one of the declared aims of our national program of study in France for some concepts: function, systems of linear equations (at grade 10), differential equation(grade 12).
- But students encounter difficulties in linking different representations and this flexibility must be developed through adequate tasks

New kind of representations with Dynamic Geometry

- Artefacts in mathematics produce representations, they are not new: paper and pencil
- But now there are artefacts offering representations of a new nature
- In dynamic geometry, representations are no longer inert but “behave” mathematically when varying
- They reflect the variations of mathematical objects
- Variation is under the control of the user (direct manipulation)
- The intermediate states are continuously made visible
- Possibility of multiple linked representations in DG
- This makes possible new ways of solving problems and new kind of problems

New kind of tasks

- Tasks centered around
  - the representations and the links between two different kinds of representations
  - and resorting to mathematical knowledge
- Interpreting the behaviors of variable representations
- Producing variable representations
- Examples
  - Moving a point in space until reaching a hidden spatial object.

Three modes of using graphical representations 1/2

- corresponding to three types of links between representation and concept
- After Chauvat 1997
- **Illustrating a concept**: a parabolic shape is a sign of a polynomial of second degree
  - The representation is a sign referring to an idea, no effective ways of operating with the representations (**ideographic mode**)
- **Operating directly on the representation**
  - The representation is an object itself on which to operate but the link with the concept is opaque (**nomographic mode**)
  - Ex: reproducing a geometric diagram by just using measures and not referring to geometrical properties

Three modes of using graphical representations 2/2

- **Operational** mode: link between the representation and the concept is used
  - Intermediate mode between the two preceding ones
  - It requires a mathematical interpretation of what is observed
  - This operational mode is involved in the example of moving a point until touching a given plane

Claim

- Dynamic representations "embarking knowledge" are tools for fostering an operational mode
  - of using representations
  - of linking representations
- Future curricula should take into account this possibility

Function and Graph

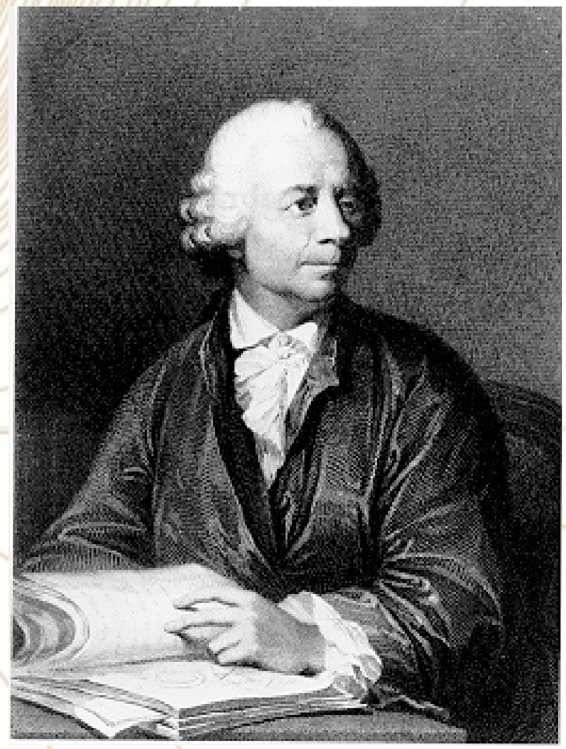
- Several kinds of representations of a function, in particular
  - Symbolic expressions
  - Graphical representations (graph)
- Each kind of representation brings to the fore different features of functions and is relevant for different uses and problems
  - The variations and extremal values are better seen on a graph
  - Numerical problems are better solved with the algebraic expression
- The correspondence between the two representations is based on the idea of representing a variable number by a variable point on an axis

The idea is not new…

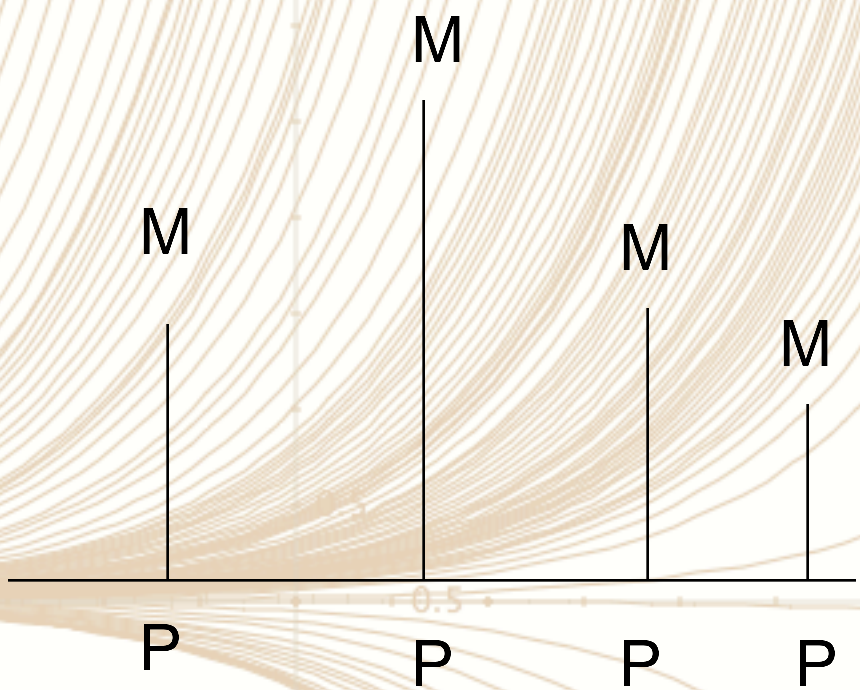
Euler

Introductio in analysis
Infinitorum
Tomus secundus

Theoriam Linearum
curvarum
Lausannae
MDCCXLIII



Euler’s thought experiment



For each value of *x*, a point *P*
For each point *P* a perpendicular segment *PM* representing *y*
When *x* is increasing from 0 to infinity, one considers all points *M* which give rise to a curve.

Students’ difficulties with graph

- However the genesis of the notion of graph is lost for students (Vinner & Dreyfus, Eisenberg, Trigueros, Markovits, Schwarz)
- The graph is just seen as an entity attached to the function (ideographic mode)
- Students have difficulties in conceiving the dual aspect of the graph
  - Set of points (x, $f(x)$)
  - A curve with geometrical properties
- A source of the problem is in paper and pencil:
  - It is impossible to draw all points (x, $f(x)$)
  - Often only some points are drawn and linked with a smooth curve, but students have no idea of what represents this curve.

Functional dependency

- The relationship of dependency between the two variables *x* and *f(x)* is not grasped by students in the paper and pencil graph:
  – It is not visible in the graph
  – The graph is static
  – The two variables play a symmetrical role
- As a result, lack of operational relationship between function and graph for high school students

DG materializing the Euler's thought experiment

- A variable number *x* is represented by a point on an axis
- In dynamic geometry the variation can be represented by motion
- In DG a variable point can leave its trace
- The graph of a function is the trajectory of point *M*
- The dual meaning of the graph is restored in the notion of trajectory

Mediation of dependency through DG

- It is possible to distinguish between the independent and dependent variables through dragging
  - The independent variable is represented by a point which is directly movable
  - The dependent variable is represented by a point which can be moved indirectly by dragging the point representing the independent variable
- In a first step, dragging is an external tool enabling students to distinguish between dependent and independent variables
- Then with the guidance of the teacher, this tool can be internalized by students who construct the concept of independent and dependent variables (semiotic mediation after Vygotsky)

From a numerical function to a geometric function

- The graph can be seen as the image of a geometric function which maps *P* to *M*
- Instead of starting from numerical functions, the idea is
  - to first introduce students with geometric functions in DG
  - then to move to numerical functions and use geometric function as a way of representing a numerical function

Teaching experiment

- Design of three sequences of activities for 15-16 year old students in France and Italy (2000-01, 2001-02, 2002-03)
  - Lycée in a suburb of Grenoble
  - Liceo Scientifico in Forte dei Marmi (Lucca)

- We wanted to investigate more deeply the process of mediation and internalization that we assumed.

Structure of the sequence

- 1) Introduction to the notion of geometric function in Cabri-geometry
  - A problem: find an unknown function
  - Imagine a new geometric function
- 2) A problem: how to represent geometrically the co-variation of two variables?
- Reading and discussing the historical solution proposed by Euler about the notion of graph of a function
- Implementing the solution of Euler in Cabri
- 3) Working on graphs with Cabri
  - Solving extrema problems

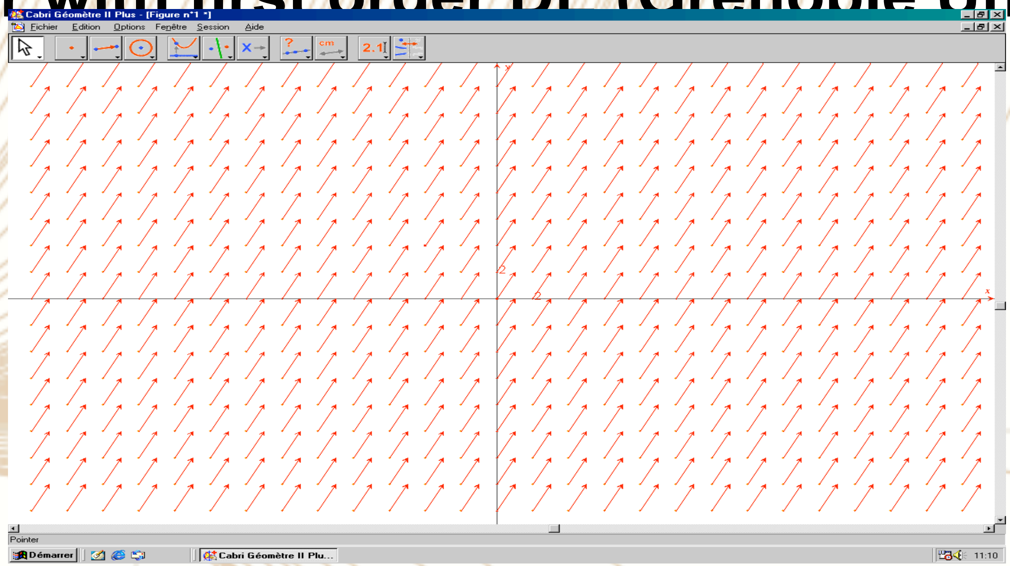
Differential equations in preservice teacher education

- Three kinds of approaches: algebraic, numerical, qualitative
- The algebraic approach prevails internationally
- Weak number of tasks including graphical representations
  - most of them require the move from symbolic expressions to graphical representations: solving equation and **then** representing the solutions

Consequences on students' abilities

Preservice teachers, students of 4th university year, already familiar with first order DE (Grenoble Univ.)

The vector field represents a vectorial function which assigns the vector (1,2) to each point (*x*,*y*) of the plane



2 students out of 56 succeeded in linking this representation to DE *y'*=2

6 students (out of 56) considered that the curves solutions are straight lines

Local study of a solution from the given DE

Let be DE $y' = y^2 - x$. C is the curve representing a solution and passing through point $M(-2,1)$. Describe the behavior of C around point M. Give an approximate value of the y coordinate of point N on C with x coordinate $x=-1,5$.

Only 4 students out of 56 described the local behavior of *C*. Among them, only 2 obtained the approximate value of *N*.

6 students (out of 56) tried to solve the DE.

Variable representations offered by DG

Variable Tangent Vector:

- One vector can generate a field of vectors
- Variations related to the algebraic properties of the function *f(x,y)* appearing in *y' = f(x,y)*

Variable Curve as a variable representation of the set of solutions:

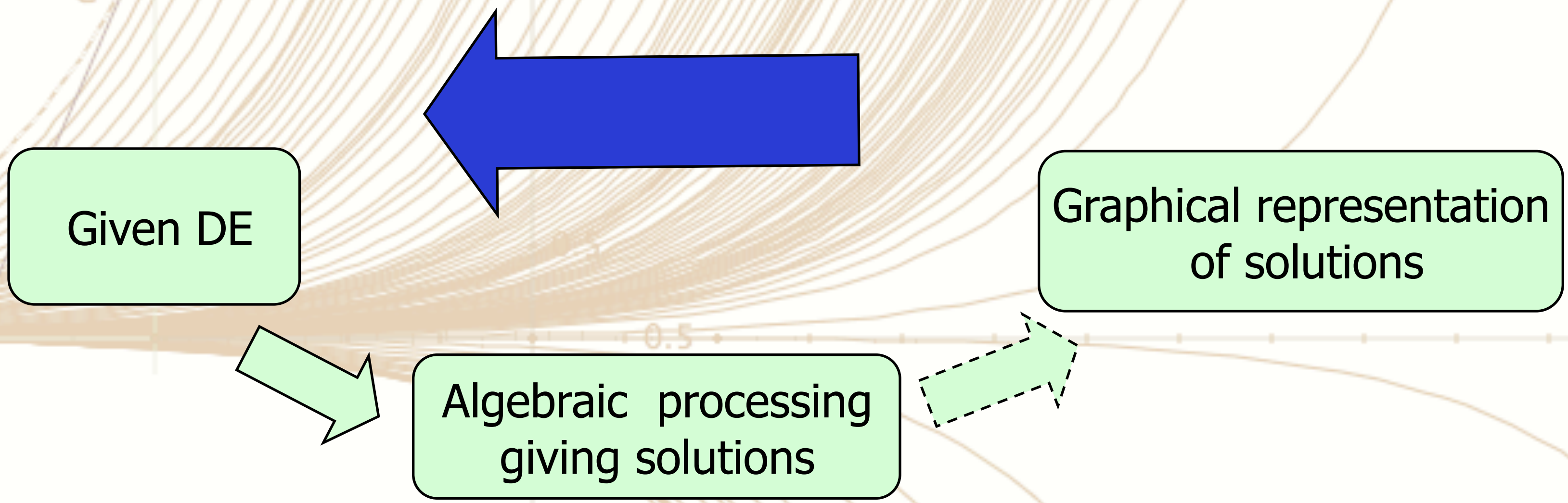
- One curve generates the family of curves representing the family of solutions
- The slope of tangent line at a point (*x*,*y*) on a curve varies as *f* (*x*,*y*)

Tasks problematizing the articulation between symbolic and graphical representations

- They require an operational mode between representations
- To a given variable vector select the corresponding DE in a list of given DE of the form $y' = f(x,y)$ and justify.

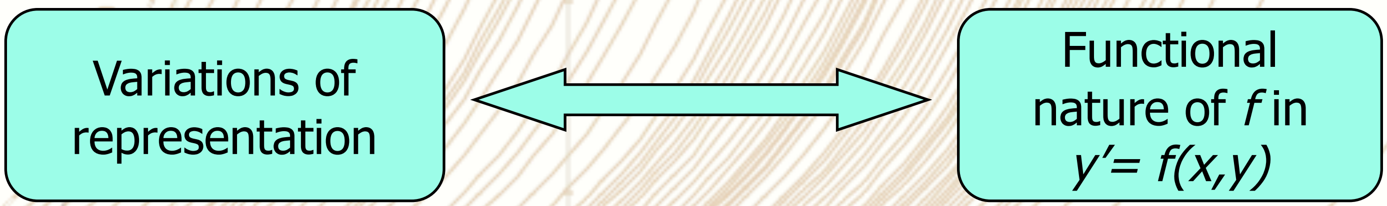
- From a given variable curve, find the DE of the generated family of curves; then give additional arguments why the obtained DE is correct.

Converse move from the algebraic approach

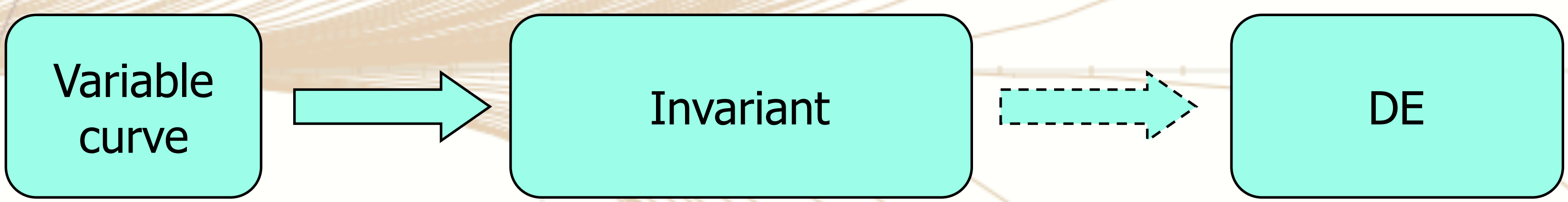


Hypotheses on the role of DG

DG offers an environment for establishing a relationship:



2. DG makes possible the problem: Curves solutions → DE



3. Justifying graphical phenomena requires articulating between graphical and symbolic representations

The tasks

- Unusual tasks but accessible to students: required mathematical knowledge is part of content taught in the previous university years
- Tasks requiring an operational mode of the representations offered by DG

From a variable curve to DE

- Given representations: a variable curve and a tangent at a variable point of the curve in Cabri II plus
- Task for the students: find the corresponding DE, in this case $y' = y$
- Three possible strategies

Algebraic strategy

- Algebraic strategy
  - Recognizing the shape of the curve as an exponential (ideographic mode)
  - Move to algebraic expression $y = e^x$
  - Algebraic processing to obtain $y' = y$
- Weak role of DG, almost no use of variation
- Use of this strategy restricted to familiar curves

Numerical strategy

- Displaying the coordinates of point *P* and the equation of the tangent line at *P*
- Varying *P* and varying the curve
- Recognizing a numerical invariant: the slope of the tangent line is “**always**” equal to the *y* coordinate of *P*
- Move from coordinate geometry to function:

Interpreting the invariant as *y*’= *y*

  – Interpreting *y*’ as a function of *x* and *y*

- Joint variation of graphical and numerical objects is essential

Graphical strategy

- Constructing the subtangent
- Varying P and the curve
- Identifying a graphical invariant: the length of the subtangent
- Expressing this invariant in an algebraic expression by giving the value *y*' to the slope of the tangent line: it requires the expectation of an expression of the form $y' = f(x,y)$

Observed strategies

- Students worked in pairs at the computer
- At the beginning
  - Algebraic strategies
    - students tried solutions $y = e^{kx}$
    - or $y' = ax + b$ or $y' = ay + b$ (influence of the equation of the tangent line)
    - or tried randomly differential equations
  - Invalidation by Cabri of the solutions: very often numerical invalidation

In a second step

- Move to the numerical strategy
- All students recognized the numerical invariant
- But most of them did not interpret it as a differential equation and stopped
- Intervention of the teacher: 1) what does express a DE? 2)can we express *y*' as a function of *x* and *y*?
- Surprise and sometimes disappointment: “That’s the differential equation!”

Window on students' conceptual difficulties

- To each curve a different DE

« Sur une courbe solution, on dirait que l'équation différentielle c'est *y'=y*. Maintenant, il faudrait peut-être tracer une autre courbe solution, et on n'aura pas forcément *y'=y* »

A pair of students wanted to add in the equation y' = y a parameter *a* depending on the curve
"May be it is y' = y (x+*a*)"

Confusion with the equation of a tangent line

- Students' knowledge is fragile and is no longer available in non usual situations
- Very procedural without links with a graphical representation

Students' Evolution

- Additional arguments: Some arguments linking algebraic expressions and graphical representations appear
  - "For the same y, all the tangent lines should be parallel"
- On the following task, from a variable tractrix determine the DE, all students had to use the graphical strategy and they succeeded

Why evolution?

- Two initial factors
  - The task itself
  - Feedback coming from the software invalidating incorrect solutions or conjectures
- Task and feedback were source of questions to teacher
- Interventions of teacher