Center for the Study of Mathematics Curriculum
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Future Curricular Trends in School Algebra and Geometry
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Closing Remarks

by Zalman Usiskin

At this point in every conference it is appropriate to thank those individuals without whose work the conference would not have taken place. There are four groups of individuals:

- the planning committee – Sarah Kasten, Ira Papick, Nathalie Sinclair, Chris Hirsch, Bob Reys, Gwen Lloyd, and Mary Ann Huntley, responsible for the selection of speakers and the program
- the local arrangements people – directed by Carol Siegel – responsible for the selection of the Field Museum as our first venue and for the wonderful refreshments
- the speakers – who have given us great ideas and inspiration
- you, the participants – without whom there is no conference

And also NSF, whose funds supported about 2/3 of this conference.

We do plan to have proceedings that every participant will receive as soon as they are completed. But don't expect them immediately; my guess is it will be in the second half of 2009. However, if you wish to see the PowerPoints of the presentations, 10 of the 15 presentations are already up on the CSMC website (www.mathcurriculumcenter.org).

This is the sixth international conference in mathematics education dealing with curriculum that we have hosted in the past 25 years. Four of these conferences have been under the auspices of the University of Chicago School Mathematics Project and the last two were run by the Center for the Study of Mathematics Curriculum. Having become emeritus at the university this past January, I think I can say with some confidence that this will be the last conference in which I have a guiding hand. In these closing remarks, I am not going to try to summarize what has happened at this conference; nor will I try to provide a perspective to match the perspectives given by Jim (Mamer), Diane (Briars), Doug (Clements), the other Jim (Fey), and just now by Bill (McCallum) and Joan (Ferrini-Mundy). I hope that you will indulge me if I use this occasion to make some personal remarks.

When I entered the University of Illinois as a freshman in 1959, I knew I wanted to teach mathematics. In the spring of my freshman year, I took my first course in mathematics education. It was basically a course in the new math, a course in the curriculum work of the University of Illinois Committee on School Mathematics. We learned about the distinction between number and numeral, that is, between the concept and what was written down. We learned about quantifiers such as *for all* and *there exists*. We learned about using properties such as commutativity and associativity and identities and the deduction of the rules for operations with positive and negative numbers. But the fundamental idea that we learned was that all the reasonably isolated skills we had learned in our high school courses could be viewed as resulting from field properties of the real numbers, and later the complex numbers.

We became imbued with the structure of mathematics and became disciples of it. We knew that we were doing mathematics that had been identified later than other mathematics. We were modernizing the curriculum with properties, logic, matrices and other algebraic concepts that had first appeared in the 19th century. We felt good that we were modernizing the curriculum.

In this university course, I was struck by the fact that there were fundamentally different ways to approach the same mathematical ideas. I wanted to know more about alternate ways to develop mathematics. I went to Harvard for my master's to study with Ed Moise, who had developed the SMSG Geometry. From the time I started teaching, I looked at different ways to approach the subject. I went to Michigan for my doctorate because they would allow me to do a dissertation involving curriculum. There I fell into the use of transformations in learning geometry.

I say I "fell" into transformations. That isn't exactly true. I was searching for a topic for my dissertation and I asked Joe Payne for some ideas and he said, "There are some people interested in using transformations in geometry. You might want to look into that." And so I did and I became enthused. The way that transformations enabled congruence, similarity, and symmetry to be developed was so elegant. And they applied to all figures, not just triangles and other polygons or circles and the common 3-D figures. My first motivation was mathematical. But something happened as Art Coxford and I were developing the course through transformations and teaching it to 10th grade students. We realized that our view of geometry was changing. We saw things in geometric figures we never saw before – properties were due to symmetry rather than to SAS, or SSS. For instance, the opposite sides of a rectangle were congruent not because we could form congruent triangles with the diagonal but because of the reflection symmetry of the figure. A rhombus was a special kind of kite – a figure that we had not seen in U.S. schoolbooks.

Having transformations in geometry not only affected our view of geometry but also our view of algebra. Graphs of functions became geometric figures, with the graph of the sine congruent to the graph of the cosine. All parabolas were similar! The cosine and sine functions could be defined on the unit circle as coordinates of the image of the point (1,0) under rotations. Matrices became important. So did groups. I was awestruck by the relationships between geometry and algebra, and how geometry now had so many applications in the later algebra courses.

It happens that we hit transformations at just the right time, and the work we did with transformations had some influence on other materials. I felt lucky to have hit this at the right time.

Shortly after this work with transformations and matrices and groups in both geometry and second-year algebra, I was encouraged by people to think about first-year algebra. Max Bell was pushing applications throughout mathematics, and I took his lead and developed a first-year algebra course through applications, including probability and statistics. Again it seems that I hit the field with the right idea at just the right time, and that work too had its influence, not just on algebra but on other courses. It certainly influenced UCSMP, which came some years later. And we felt the UCSMP curriculum developed before the NCTM curriculum standards influenced those standards.

About five years after this algebra course through applications was developed, about 25 years ago, UCSMP began. From the very beginning, we were committed to using the latest in technology. I spoke about CAS in a major talk at the NCTM annual meeting in 1983; I thought CAS would start appearing in school curricula in a couple of years. It has taken longer. Only now are we seeing school curricula appear with a significant amount of CAS work.

Through the years many people told me that they felt I had some sort of knack for foreseeing what happens in curriculum. Early in my career I thought it was luck. Later in my career I started wondering whether the people were right – maybe I had some sort of special knack for latching onto ideas that later became used.

But now, looking back at an entire career, I can see that it was neither luck nor some special knack. I was in the *field* of mathematics curriculum and at the places where cutting-edge work was being done or being considered. In this field, many others were doing the same thing and had been working long before I did. Transformations had been used in Europe ever since Felix Klein almost 100 years before Art and I used them in the U.S., and Zoltan Dienes in Hungary and Gustave Choquet in France were talking about the use of transformations with students before we were working with them. Howard Fehr and Jim Fey were working simultaneously with transformations at Columbia University with the Secondary School Mathematics Curriculum Improvement Study. (Paul) Kelly and (Norm) Ladd had written a geometry book with a long chapter on transformations.

In applications of mathematics, Max Bell and Henry Pollak and others had been working in applications years before I got into that arena, and Sol Garfunkel went into it with a vengeance with COMAP. Jim Fey and Kathy Heid were working with CAS. Many others have been involved in the notion that you could approach mathematics differently and change how students view the subject, including Chris Hirsch and Glenda Lappan and Bob and Barbara Reys and many others who have been working in the field for over a quarter of a century, and so many others who joined us in the 1990s with the NSF-supported projects.

Through all these years, the University of Chicago fortunately provided an environment that supported this curriculum work. Max Bell and I were able to use our work to obtain tenure and promotions to full professors at a time when comparable institutions – Harvard, Yale, Princeton, and Stanford – had no mathematics education at all.

In the early 90s, NSF played an important role in the field of curriculum development. By funding over a dozen multi-year curriculum projects, and by encouraging – well, actually, forcing – the people from those projects to get together in what were known as the Gateways Project conferences, NSF caused universities to realize that the study of mathematics curriculum was a viable research area. By funding full curricula, it forced the developers to have to deal with the complexities of the mathematics curriculum and the intersections of curriculum with policy, schools, classrooms, and students. By establishing the Arc, Show-Me, and Compass Centers to get the word out about the curricula, NSF may have been a little over-enthusiastic about trumpeting its own curricula, but it again helped to bring the field together by forcing everyone to realize that curriculum development is a serious field of study driven by what is best for students and teachers in various places and what will most likely improve mathematics education.

Today, however, we have seen disturbing developments. Despite the very public statements of the U.S. Department of Education of the importance of well-constructed comparative curriculum research, there does not seem to be much of an effort to support the basic curriculum development work that leads to that research. Indeed, sometimes people who use different approaches to school mathematics, we people who do curriculum development, are looked at as if we are in alternative medicine, doing unsanctioned behavior that will surely ruin the nation's students. The notion that *new* curricula must pass some litmus test before they can be used, but older curricula that have failed do not have to pass any test, is a mechanism to quash any sort of curriculum change.

We cannot wait for basic research to underlie everything we do in education. If we did, we would have nothing to teach. For instance, there is virtually no research in the learning of statistics but students must learn statistics. And the gold standard has a moral aspect with respect to technology. Medical studies are often stopped in the middle when it is clear that the new method is so much better than the old. This is the case with powerful technology. It is not fair to one group to give them technology and not give that technology to another group. Similarly, we cannot have research comparing students who are taught statistics (or any other new topic) with those who are not, because it is not fair to give a test to students over content they have not studied. We have to do the best we can, with curricula that are comfortable for students and teachers, and curricula that show students can learn these ideas. Fortunately, the choice of what to teach is a matter of belief rather than research. The recent report on school algebra of the National Mathematics Advisory Panel makes that patently clear.

That panel did not understand many things. But fundamentally, what they did not realize is that the mathematics curriculum is a living organism that moves in reaction both to its heredity and its environment. Of its heredity, one parent might be said to be pure mathematics and statistics and computer science – and the other parent is applied mathematics – consumer mathematics and quantitative literacy and the nonacademic uses of mathematics in business and everyday life. The environmental influence on mathematics curriculum is the students and the teachers and the schools and the communities in which learning takes place. Those of us who work in curriculum naturally look at what is going on and try to nourish this organism as we think best. In trying to create the healthiest possible organism, we must consider both heredity and environmental factors. It is natural for many of us to be thinking somewhat alike but not exactly alike, just as medical researchers looking at a disease may have similar but different plans of attack. We usually see parts of the organism as needing help and other parts that are best left alone.

The two notions – first that the mathematics curriculum is a living organism in constant need of examination for its wellness and sickness, and second that the study and work of that organism is a field – and not just people's hobbies or happenstances of current events – can be seen as *the* construct underlying the existence of the Center for the Study of Mathematics Curriculum. This is why the CSMC's work is critical for the future of mathematics education in our country, why we feel it is so important to develop students whose doctoral study involves concentrated work in mathematics curriculum.

It is the total disdain for this work that underlies what disturbs me and some others of us so much in the report of the National Mathematics Advisory Panel on school algebra and – perhaps less so, but for me, just as significant - also about the NCTM Focal Points.

The panel's view of algebra goes up only one parent of the hereditary tree of mathematics, and only one grandparent, and perhaps only one great-grandparent. As I quipped in one of the concurrent sessions (at this conference), it should not have been called the National Mathematics Advisory Panel, but the National Paper-and-Pencil Manipulative Algebraic Skills Advisory Panel. Then its report would make sense. But as is, the report does not make any sense, for it completely ignores all of the other heredity of algebra. And it also completely ignores the change in environment and everything that has been done regarding the algebra curriculum in the past forty years because of that change in environment.

Regarding the NCTM Focal Points, returning to the living organism analogy, we know everyone should eat fruits and vegetables, but do not know enough to know whether having particular fruits at particular times is better than others. The Focal Points are simply too specific a diet. They constrain rather than expand the repertoire of concepts and approaches that are in some sense sanctioned by specifying too arbitrarily the age at which certain ideas should be encountered. The work in this conference shows us that the age at which serious work in algebra should begin is not at all settled. The fact that various states have differed on when they should introduce or master many topics in arithmetic or geometry is evidence that setting specific grade levels for many topics is rather arbitrary. Furthermore, by doing so one introduces serious gaps. For instance, there is no geometry in grade 6 in the Focal Points. Not in the focal points or in the accompanying ideas. None at all. Is there something about that age – age 11 and 12 for most students – that would indicate geometry should not be studied? Of course not. The Standards of 1989 and PSSM of 2000 were much more like broad guidelines (thank you Joan (Ferrini-Mundy)), with the appropriate balance of latitude and specificity. Let us hope that NCTM steps back in time and realizes the errors of its recent ways. As Diane Resek suggested yesterday in a session, the 1989 Curriculum Standards may be the best document we have created so far.

This conference was titled "Future Curricular Trends in School Algebra and Geometry". Did we pick the right trends? Frankly, we picked safe trends, because the four ideas that we chose for discussion at this conference are not new ideas. Algebra used to be a college subject. The vestige of this is in the fact that there is a course in community colleges called "college algebra" that has the same content that we have in advanced algebra and precalculus mathematics. Statistics used to be taught first in colleges. Geometry used to be absent from early elementary school. Over 40% of 8th graders in the U.S. now are taking an algebra course. Algebra is being taught earlier and earlier, and the work described here is continuing a multi-century trend. It is a safe prediction that what we now call "early algebra" will in the far future not be considered early.

Our second theme, CAS, is relatively new in that computer-algebra systems were first developed in the early 1970s. But the notion of developing new algorithms for solving mathematics problems is as old as mathematics itself. The Babylonians and Egyptians described algorithms 3000 years ago; the Chinese developed what we call the Chinese remainder theorem to solve problems that today we call systems of equations in modular arithmetic. In the middle ages there was a conflict between the abacists, those who did arithmetic on the abacus, and the algorithmists, those who used paper-and-writing-implement (there were no pencils at first) algorithms that were new for the day. Newton's method for solving polynomials and the calculus were valuable because they provided powerful algorithms for solving many questions about curves and physics. One use of CAS, to provide easy and automatic ways of solving hitherto very difficult problems, simply follows what has been perhaps the main purpose over the years – to solve problems in an efficient way. This, too, is a safe theme. Another use of CAS was not discussed much at this conference – its value as an instructional tool particularly in helping slower students learn algebra. CAS is here to stay and the only question is the speed with which curricula will adapt to handle the new and powerful ways it gives us to look at mathematics and solve problems.

Our third theme, 3-D geometry, is in some sense the most enigmatic of mathematics curriculum themes, but it, too, is nothing new. There is solid geometry in Euclid's *Elements* and trigonometry was first developed from the celestial sphere, not the plane. Our problem has always been to represent the 3-D world on paper. We still have not solved that problem, but new technology shows us that we can represent our 3-D world quite nicely on a 2-D screen by manipulating objects in space. This may be a virtual environment, but isn't paper-and-pencil also a virtual environment? When Hilbert said that mathematics dealt with the marks and symbols drawn on paper, could we not interpret that as saying that mathematics is itself a virtual environment?

The fourth theme, the integration and linking of algebra and geometry, also has ancient origins. The geometric approach to number and what is today's quadratic equations found in Euclid's *Elements*, the analytic geometry of Fermat and Descartes, the calculus of Newton and Leibniz all bring both algebra and geometry to play. Bringing these subjects together is bringing them *back* together after a hiatus of a couple of hundred years while their foundations were being strengthened logically.

Thus it is not much of a prediction to say that the themes of this conference will be significant in keeping the organism healthy in the years to come. Algebra will become a staple of elementary school mathematics, maybe not in the next few years, but surely in the next few centuries. It is as inevitable as arithmetic's moving down over time from college to elementary school. 3-D geometry will come back into the curriculum through technology if for no other reason than the world is still, for almost all practical purposes, 3-dimensional. Geometry and algebra will become more and more intertwined as we see that each represents the other and have the technology to show the simultaneous representations. CAS will revolutionize the learning of algebraic algorithms in the same way that paper-and-pencil technology revolutionized the learning of arithmetic. Some of these things may happen in just a few years; others may take centuries. But they are inevitable, because organisms do not just live, they evolve. We in the Center for the Study of Mathematics Curriculum hope that you have enjoyed looking at these forces moving the living organism of mathematics curriculum.

We hope that you have found the past two days here to be stimulating and informative and wish you safe trips back home. Please take a snack for the road and please remember to complete the evaluation forms. And again, thank you for coming and contributing to the conference.